Approach toward the anomalous diffusion in the soil by homogenization

Masaaki Uesaka
(The University of Tokyo)

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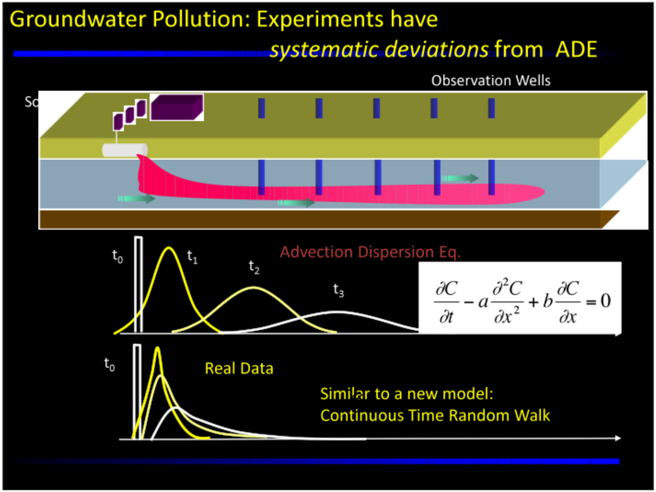
Joint Project with..

- Dept. Math. Sci. of The University of Tokyo
- Faculty of Engineering of The University of Tokyo
- Risk Management Dept. of Tsukuba University
- Nippon Steel Cooperation

Outline

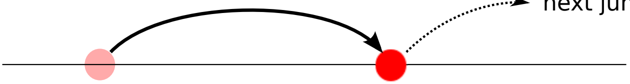
Diffusion in the soil is...

- **important in environmental problem.**
  Prediction of the contamination: chemical contaminant, radioactive material
- **multi-scale problem**
  - Field: 10km–100km
  - Soil grain and pore: about 100μm
- **anomalous diffusion**
  The diffusion cannot be described by the ordinary advection-diffusion equation.


Groundwater Pollution: Experiments have
systematic deviations from ADE
Observation Wells
t_0
t_1
t_2
t_3
Advection Dispersion Eq.
$\frac{\partial C}{\partial t} - a\frac{\partial^2 C}{\partial x^2} + b\frac{\partial C}{\partial x} = 0$
t_0
Real Data
Similar to a new model:
Continuous Time Random Walk

Approach to the anomalous diffusion

Continuous Time Random Walk Model



- Two key probabilistic distribution functions(pdfs):
  - $\phi(x)$: pdf of the jumping length
  - $w(t)$: pdf of the waiting time
- If $w(t) \sim e^{-t/\tau}$(Exponentially) and ϕ is a Gaussian distribution, then the master equation of this random walk is ordinary diffusion equation.
- cf. Metzler R and Klafter J, THE RANDOM WALK'S GUIDE TO ANOMALOUS DIFFUSION: A FRACTIONAL DYNAMICS APPROACH, Physics Reports 339(2000) 1–77

CTRW to fractional diffusion equation

CTRW→**Fractional Diffusion Equation**
Let $w(t) \sim t^{-\alpha}$ and $\phi(x)$ is a Gaussian distribution. Then the master equation is a fractional differential equation:

$$\partial_t^\alpha u(x,t) = D(x)\Delta u(x,t),$$

where

$$\partial_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\mathrm{d}f}{\mathrm{d}t}(\tau)\,\mathrm{d}\tau.$$

This equation shows at most polynomial-order decay:

$$u(x,t) \sim t^{-\alpha}$$

cf. Sakamoto and Yamamoto (2011, JMAA).

Problem

- CTRW & fractional diffusion equation leads to the anomality but does not tell us the mechanism of anomality.
- This mechanism depends on the micro-structure of the soil.

Problem

What structure leads to the historical effect like

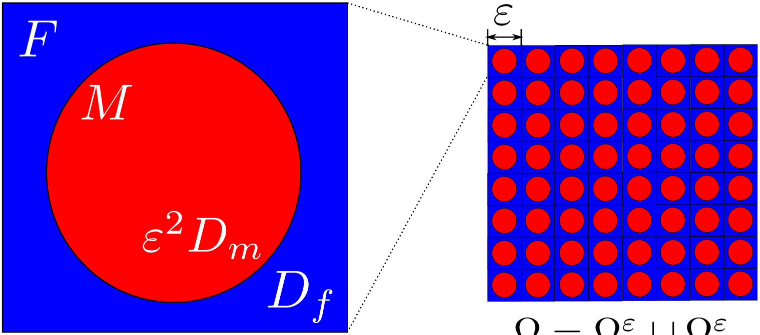
$$\partial_t^\alpha u(x,t) = D(x)\Delta u(x,t),$$

where

$$\partial_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\mathrm{d}f}{\mathrm{d}t}(\tau)\,\mathrm{d}\tau ?$$

Homogenization of highly heterogeneous media

Auriault and Lewandowska(1995), (Allaire (1992))
Diffusion in the composite porous media



$$Y = (0,1)^n \qquad \Omega = \Omega_f^\varepsilon \cup \Omega_m^\varepsilon$$

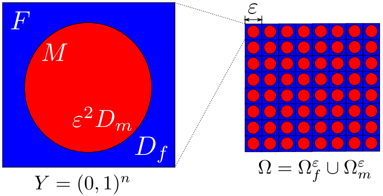
Homogenization of highly heterogeneous media

Auriault and Lewandowska(1995), (Allaire (1992))
Diffusion in the composite porous media

$$\begin{cases} \partial_t c^\varepsilon - \nabla \cdot (D^\varepsilon(\cdot/\varepsilon)\nabla c^\varepsilon) = 0 & \text{in } \Omega \times (0,T); \\ c^\varepsilon(x,0) = c_0(x); & \text{in } \Omega; \\ \partial_\nu c^\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where

$$D^\varepsilon(y) = D_f(y)\mathbf{1}_F(y) + \varepsilon^2 D_m(y)\mathbf{1}_M(y).$$



Homogenization of highly heterogeneous media

c^ε converges to the solution of following homogenized problem:

homogenized problem

$$\begin{cases} \partial_t c^* - \int_0^t K(t-\tau)\partial_\tau^2 c^* \,\mathrm{d}\tau = \nabla\cdot\left(D^{\mathrm{eff}}\nabla c^*\right) & \text{in } \Omega\times(0,T); \\ c^*(x,0) = c_0(x) & \text{in } \Omega; \\ \partial_\nu c^* = 0 & \text{on } \partial\Omega\times(0,T), \end{cases}$$

where D^{eff} and K are determined by D_m, D_f and M. Especially $K(t)$ is an inverse Laplace transform of the function $f(p) = \frac{1}{p}\int_M k(y;p)\mathrm{d}y$, where $k(y;p)$ is a solution of

$$\begin{cases} \nabla_y\cdot\left(D_m\nabla_y k(\cdot;p)\right) = p\left(k(\cdot;p)-1\right) & \text{in } M; \\ k(\cdot,p) = 0 & \text{in } \partial M. \end{cases}$$

ε^2 is essential!

Suppose that

$$D^\varepsilon(y) = D_f(y)\mathbf{1}_F(y) + \varepsilon^{\mathbf{q}} D_m(y)\mathbf{1}_M(y).$$

for $q > 2$. Then c^ε converges to the solution of following homogenized problem:

homogenized problem

$$\begin{cases} |F|\partial_t c^* = \nabla \cdot \left(D^{\mathrm{eff}} \nabla c^*\right) & \text{in } \Omega \times (0,T); \\ c^*(x,0) = c_0(x) & \text{in } \Omega; \\ \partial_\nu c^* = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where D^{eff} is determined by D_m, D_f and M. We denote by $|F|$ the volume of F.

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No memory term appears in homogenized equation.

Some double-porosity model

Amaziane, Goncharenko and Pankratov(2005, Euro. Jnl of Appl. Math.)
Flow in the porous media with thin fissure

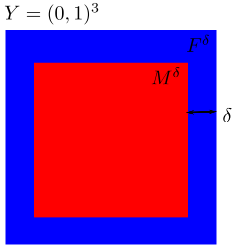
$$\begin{cases} \omega^{\varepsilon,\delta}\partial_t u^{\varepsilon,\delta} - \nabla\cdot(K^{\varepsilon,\delta}\nabla u^{\varepsilon,\delta}) = f^{\varepsilon,\delta} & \text{in } (0,T)\times\Omega; \\ \partial_\nu u^{\varepsilon,\delta} = 0 & \text{on } (0,T)\times\partial\Omega; \\ u^{\varepsilon,\delta}(0,x) = 0 & \text{in } \Omega. \end{cases} \qquad (1)$$

where

$$K^{\varepsilon,\delta}(x) := (\varepsilon\delta)^2 k_f \mathbf{1}_m^{\varepsilon,\delta}(x) + k_f \mathbf{1}_f^{\varepsilon,\delta}(x)$$

$$\omega^{\varepsilon,\delta}(x) := \omega_m \mathbf{1}_m^{\varepsilon,\delta}(x) + \omega_f \mathbf{1}_f^{\varepsilon,\delta}(x).$$

What happens as $\varepsilon, \delta \to 0$?



Fractional derivative appears!

Then for any $t \in (0, T)$, the following statements hold:

1. $$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left\| \omega_{\varepsilon,\delta} u^{\varepsilon,\delta}(t) - t f_m \right\|^2_{L^2(\Omega)} = 0;$$

2. the function $u^{\varepsilon,\delta}|_{\Omega_f^{\varepsilon,\delta}}$ $L_{\varepsilon,\delta}$-converges to u^* which is the solution of

$$\begin{cases} \omega_m \partial_t u^* - \nabla \cdot ((2k_f/3)\nabla u^*) = f_0 + f_m + S(u^*) & \text{in } (0, T) \times \Omega; \\ \partial_\nu u^* = 0 & \text{on } (0, T) \times \partial\Omega; \\ u^*(0, x) = 0 & \text{in } \Omega, \end{cases}$$

where

$$S(u^*) := -\frac{2\sqrt{k_f \omega_m}}{\sqrt{\pi}} \int_0^t \frac{u^*(x,\tau)}{\sqrt{t-\tau}} \, \mathrm{d}\tau + 4 f_m(x) \sqrt{\frac{t k_f}{\pi \omega_m}}.$$

Fractional derivative appears!

The function $u^{\varepsilon,\delta}|_{\Omega_f^{\varepsilon,\delta}}$ $L_{\varepsilon,\delta}$-converges to u^* which is the solution of

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Definition. Let $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ be a sequence with $u^{\varepsilon,\delta} \in L^2(\Omega_f^{\varepsilon,\delta})$ for any $\varepsilon, \delta > 0$. $(u^{\varepsilon,\delta})$ is said to $L_{\varepsilon,\delta}$-converge to a function $u \in L^2(\Omega)$ if

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{|\Omega_f^{\varepsilon,\delta}|}\|u^{\varepsilon,\delta} - u\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} = 0.$$

Summary

- Homogenization of highly heterogeneous media leads to the historical effect term.
- Characteristic scale relation between
  - the length scale of the structure and
  - the ratio of the diffusion coefficient

  exists.
- Can we find such kind of heterogeneity in the soil structure?
- Other mechanisms influence the anomality:
  - Adsorption by the grain
  - Complicated path in the grain structure

Thank you for your attention!!