CoMFoS11, Continuum Mechanics Focusing on Singularity
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DEVELOPMENT OF CRACKED SHELL ELEMENT USING PARTICLE DISCRETIZATION SCHEME

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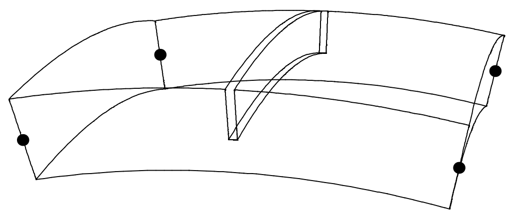
BACKGROUND

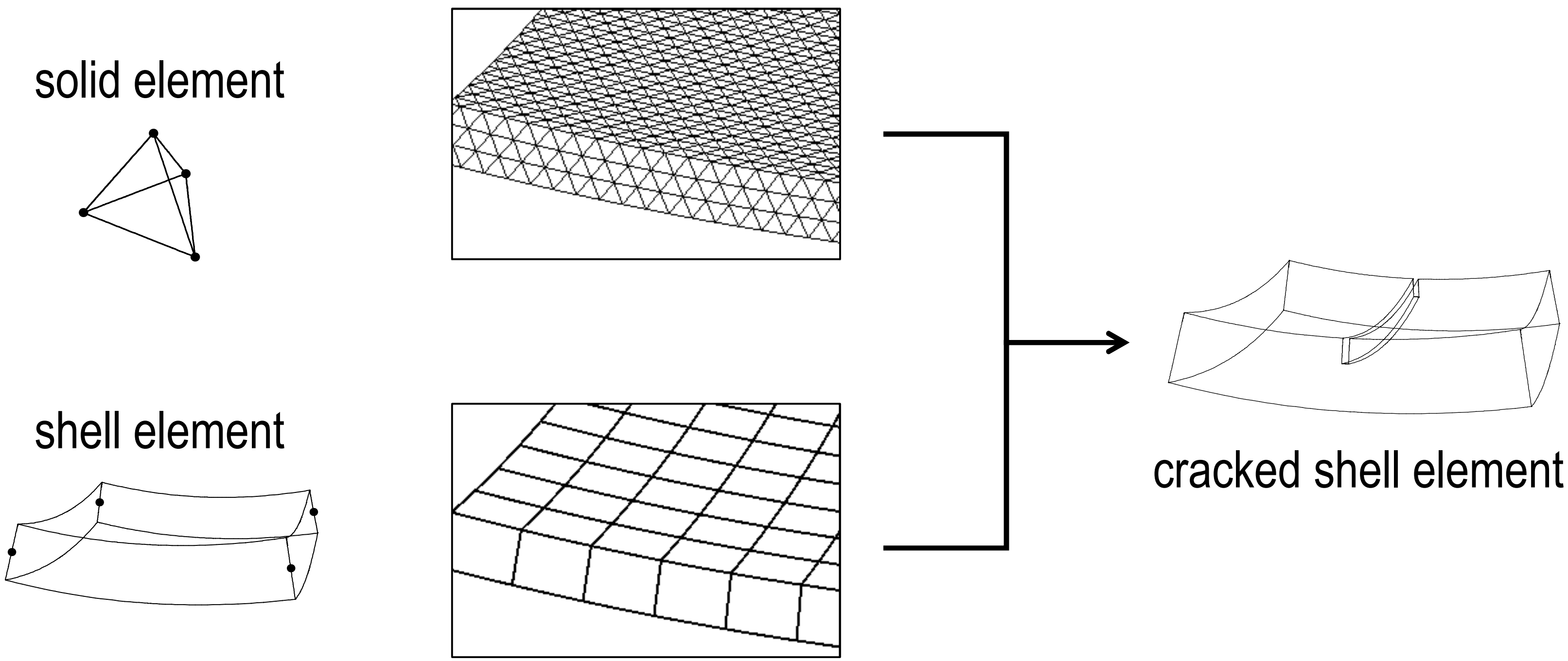
- Need for analyzing effects of surface crack on shell structure behaviors
  - estimate of stiffness reduction
  - estimate of vibration characteristic change
- Solid Element FEM Analysis
  - able to analyze crack initiation/propagation
  - computationally expensive
- Need for cracked shell element
  - crack of simple configuration



4 node 20 DOF cracked shell element

SOLID VS SHELL

element type	computational cost	cracked structure
solid	large	possible
shell	small	impossible



DIFFICULTIES IN DEVELOPMENT

◆Crack modeling

- displacement discontinuity
- singularity at crack tip

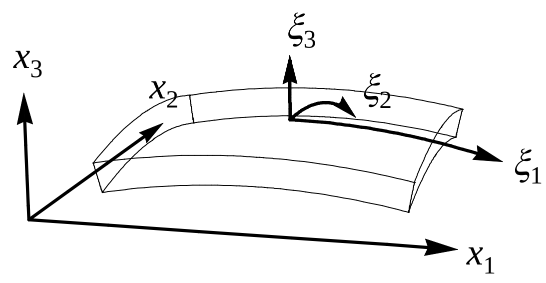
finest discretization in terms of smooth shape functions or employment of special function

➥ use Particle Discretization Scheme (PDS)

◆Complexity of shell theory

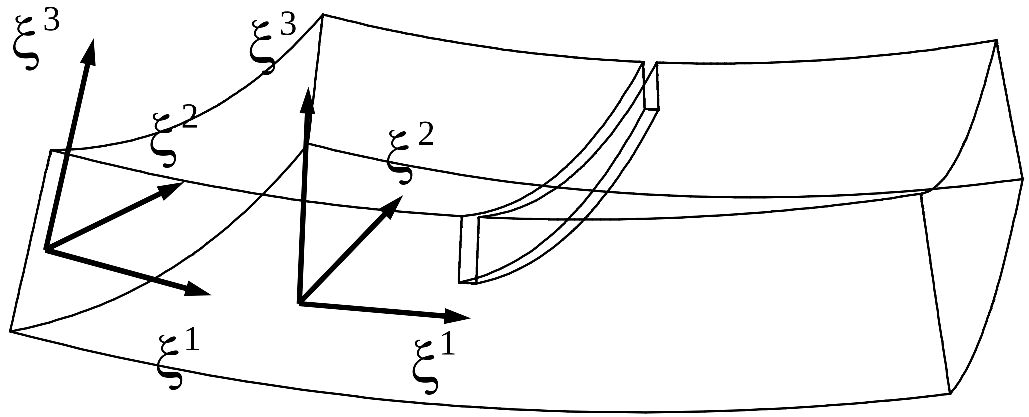
- structure in curvilinear coordinate system
- straightforward but tedious manipulations

➥ use Mathematica



Curved shape,
Curvilinear coordinates

CURVILINEAR COORDINATE



$$\frac{\partial u^i \boldsymbol{e}_i}{\partial \xi^j} = \frac{\partial u^i}{\partial \xi^j}\boldsymbol{e}_i + u^i \frac{\partial \boldsymbol{e}_i}{\partial \xi^j}$$

$$= \frac{\partial u^i}{\partial \xi^j}\boldsymbol{e}_i + u^i \Gamma^j_{ik} \boldsymbol{e}_k$$

Christoffel's symbol

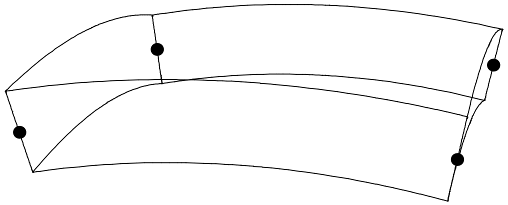
$$\Rightarrow \quad u^i_{;j} = u^i_{,j} + \Gamma^i_{jk} u^k$$

PROCEDURES

1. Derive shell functional from elastic body Lagrangian

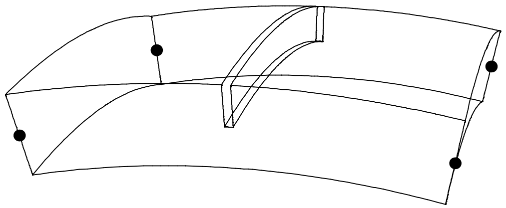
$$L^M(U, V, W, \varepsilon^i{}_j, \sigma_i{}^j)$$

2. Develop shell element using PDS



$$[K_{PDS}]$$

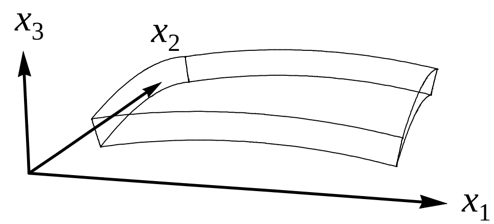
3. Develop cracked shell element using PDS



$$[K_{PDScrack}]$$

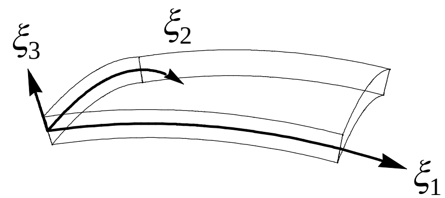
SHELL FUNCTIONAL

- Start from Lagrangian of elastic continuum in Cartesian coordinates



$$L^B(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = \int_B \frac{1}{2}\varepsilon_{ij}c_{ijkl}\varepsilon_{kl} + \sigma_{ij}(u_{i,j} - \varepsilon_{ij})d\mathbf{x}$$

- Coordinate transformation to curvilinear coordinates
  - Covariant derivatives and Christoffel's symbols:

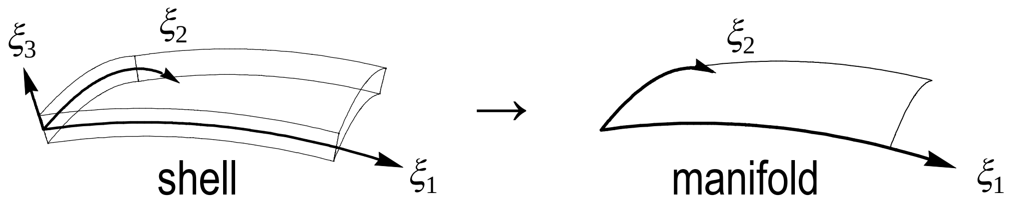


$$u^i_{;j} = u^i_{,j} + \Gamma^i_{jk}u^k$$

$$L^{B'}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = \int_{B'} \left\{\tfrac{1}{2}\varepsilon^i{}_j c_i{}^{jl}{}_k \varepsilon^k{}_l + \sigma_i{}^j(u^i_{;j} - \varepsilon^i{}_j)\right\} J d\boldsymbol{\xi}$$

  - Too many factors: expanded $L^{B'}$ becomes 171 pages in Mathematica output

2D MANIFOLD ASSUMPTIONS



1. Functions are approximated at mid-plane of shell ($\xi_3 = 0$)
   ➥Expanded $L^{B'}$ reduces from 171pages → 7pages (4%)

$$J(\xi_1,\xi_2,\xi_3) = J(\xi_1,\xi_2,0), \quad \Gamma^i_{jk}(\xi_1,\xi_2,\xi_3) = \Gamma^i_{jk}(\xi_1,\xi_2,0)$$

$$\mathbf{e}_i(\xi_1,\xi_2,\xi_3) = \mathbf{e}_i(\xi_1,\xi_2,0), \quad \mathbf{e}^i(\xi_1,\xi_2,\xi_3) = \mathbf{e}^i(\xi_1,\xi_2,0)$$

2. Form of displacement functions is assumed

$$\begin{Bmatrix} u_1(\xi_1,\xi_2,\xi_3) \\ u_2(\xi_1,\xi_2,\xi_3) \\ u_3(\xi_1,\xi_2,\xi_3) \end{Bmatrix} = \begin{Bmatrix} U(\xi_1,\xi_2) - \xi_3 W_{,1}(\xi_1,\xi_2) \\ V(\xi_1,\xi_2) - \xi_3 W_{,2}(\xi_1,\xi_2) \\ W(\xi_1,\xi_2) \end{Bmatrix}$$

$$L^{B'} \downarrow L^M$$

shell functional

VERIFICATION

- Comparison of governing equations with shell theory
  - derivation of governing equation by taking variation of L^M

$$L^B$$

$\downarrow$ coordinate transformation

$$L^{B'}$$

$\downarrow$ manifold assumption

$$L^M$$

$\swarrow$ $\delta L^M = 0$

governing equation

- Coincidence of Timoshenko's solution for cylindrical shell

ORTHOGONAL CASE 1/2

1. Lagrangian of 3D elastic body

$$J(\mathbf{u},\boldsymbol{\epsilon},\boldsymbol{\sigma}) = \int_V \frac{1}{2}\boldsymbol{\epsilon}:\mathbf{c}:\boldsymbol{\epsilon} - \boldsymbol{\sigma}:(\boldsymbol{\epsilon} - \nabla\mathbf{u})\,\mathrm{d}v$$

2. Curvilinear coordinate and manifold assumption

$$u_a = U_a(\xi_1,\xi_2) - \xi_3 W_{,a}(\xi_1,\xi_2), \quad u_3 = W(\xi_1,\xi_2)$$
$$\epsilon_{ab} = \sum_{p=0}^{1} \xi_3^p E_{ab}^p(\xi_1,\xi_2), \quad \sigma_{ab} = \sum_{p=0}^{1} \xi_3^p \Sigma_{ab}^p(\xi_1,\xi_2)$$

3. Shell functional

$$J = \int_M \Big(\frac{1}{2}E_{ab}^0 c_{abcd} E_{cd}^0 + \frac{1}{2} I\, E_{ab}^1 c_{abcd} E_{cd}^1 - \Sigma_{ab}^0(E_{ab}^0 - U_{a;b}) - I\,\Sigma_{ab}^1(E_{ab}^1 - (W_{,ab} + \Gamma_{abc}W_{,c}))\Big)\, J\,\mathrm{d}s$$

ORTHOGONAL CASE 2/2

4. Solution for stress and strain

$$\Sigma^p_{ab} = c_{abcd} E^p_{cd} \quad (p = 0, 1),$$
$$E^0_{ab} = \mathrm{sym}\{U_{a;b}\}, \ \ E^1_{ab} = -W_{,ab} - \Gamma_{abc} W_{,c}$$

5. Shell functional for displacement

$$J = \int_M \Big(\frac{1}{2} U_{a;b} c_{abcd} U_{c;d}$$
$$+ \frac{1}{2} (W_{,ab} + \Gamma_{abe} W_{,e}) c_{abcd} (W_{,cd} + \Gamma_{cdf} W_{,f}) \Big) \ J \, ds$$

6. Governing equation for displacement

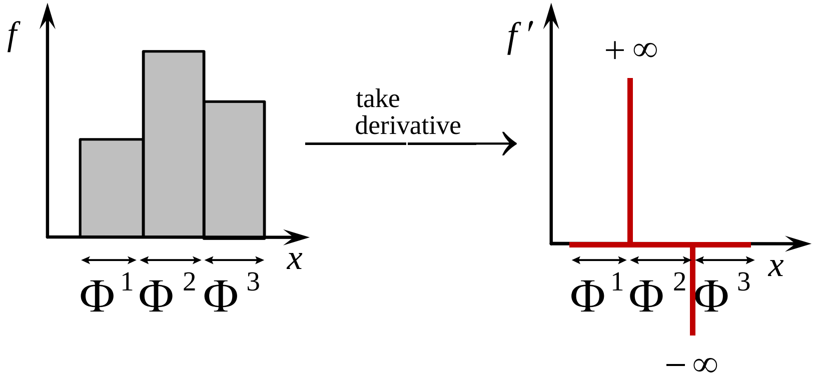
$$(c_{abcd} U_{c;d})_{,b} - \Gamma_{bea} c_{becd} U_{c;d} \ J = 0,$$
$$(c_{abcd} (W_{,cd} + \Gamma_{cdf} W_{,f}) \ J)_{,ab}$$
$$-(\Gamma_{abe} c_{abcd} (W_{,cd} + \Gamma_{cdf} W_{,f}) \ J)_{,e} + \Gamma_{ab3} c_{abcd} U_{c;d} \ J = 0$$

1D PDS

- Discretize functions using characteristic functions

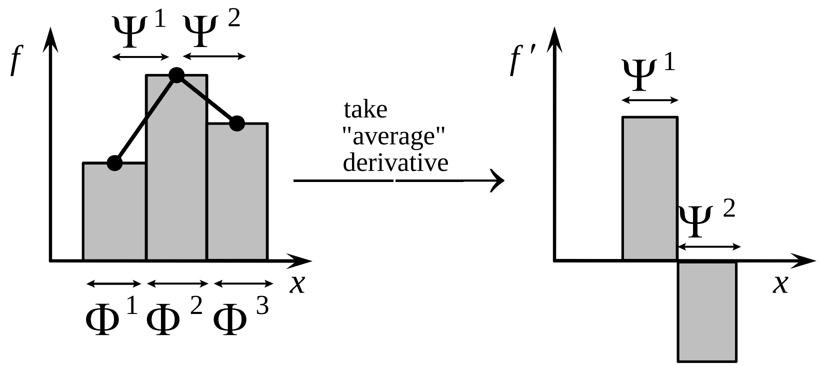
$$f(\mathbf{x}) = \sum_{\alpha} f^{\alpha} \phi^{\alpha}(\mathbf{x})$$

$$\phi^{\alpha}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Phi^{\alpha} \\ 0 & \mathbf{x} \notin \Phi^{\alpha} \end{cases}$$



- Discretize derivatives separately

$$f_{,i}(\mathbf{x}) = \sum_{\beta} g_i^{\beta} \psi^{\beta}(\mathbf{x})$$

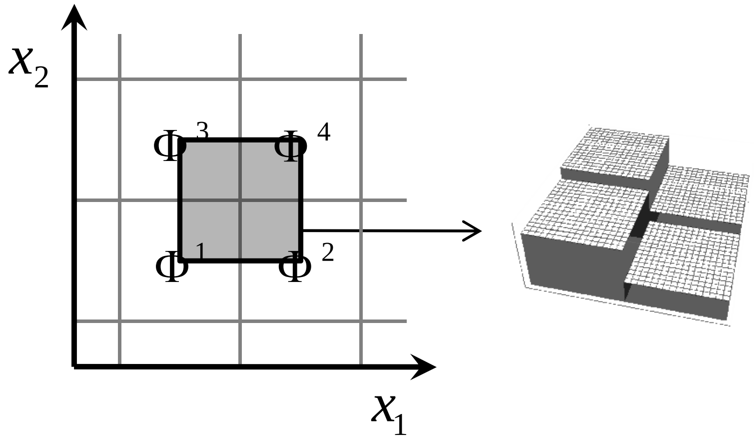


PDS

- Discretize functions using characteristic functions

$$f(\mathbf{x}) = \sum_{\alpha} f^{\alpha} \phi^{\alpha}(\mathbf{x})$$

$$\phi^{\alpha}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Phi^{\alpha} \\ 0 & \mathbf{x} \notin \Phi^{\alpha} \end{cases}$$

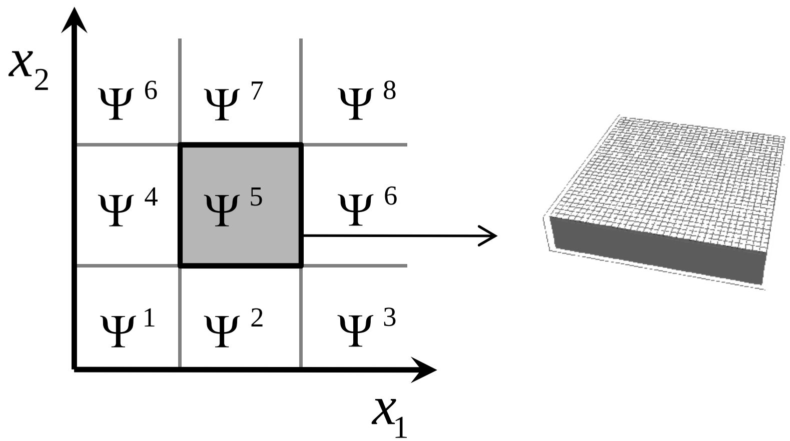


➥ suited for discretization of crack

➥ divergence of derivative at boundary

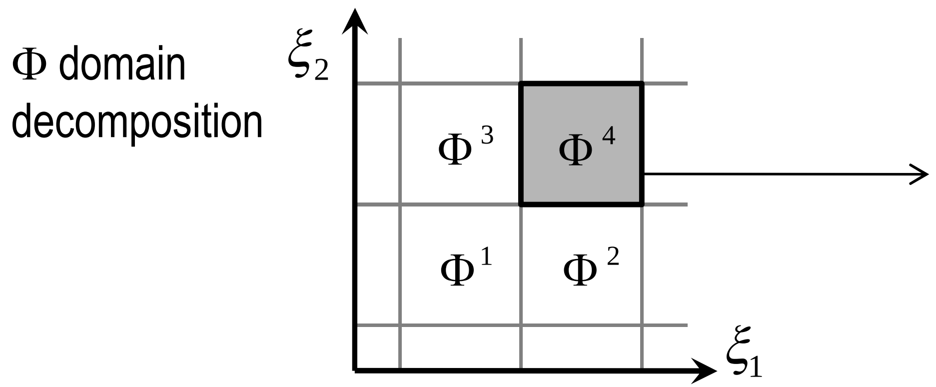
- Discretize derivative separately

$$f_{,i}(\mathbf{x}) = \sum_{\beta} g_i^{\beta} \psi^{\beta}(\mathbf{x})$$



PDS WITH POLYNOMIALS

- Characteristic function regarded as 0th order polynomial



extend

$$f(\mathbf{x}) = \sum_{\alpha} f^{\alpha} \phi^{\alpha}(\mathbf{x})$$

$$f(\mathbf{x}) = \sum_{\alpha} \left(f^{\alpha} + f_1^{\alpha} x_1 + f_2^{\alpha} x_2\right) \phi^{\alpha}(\mathbf{x})$$

polynomials

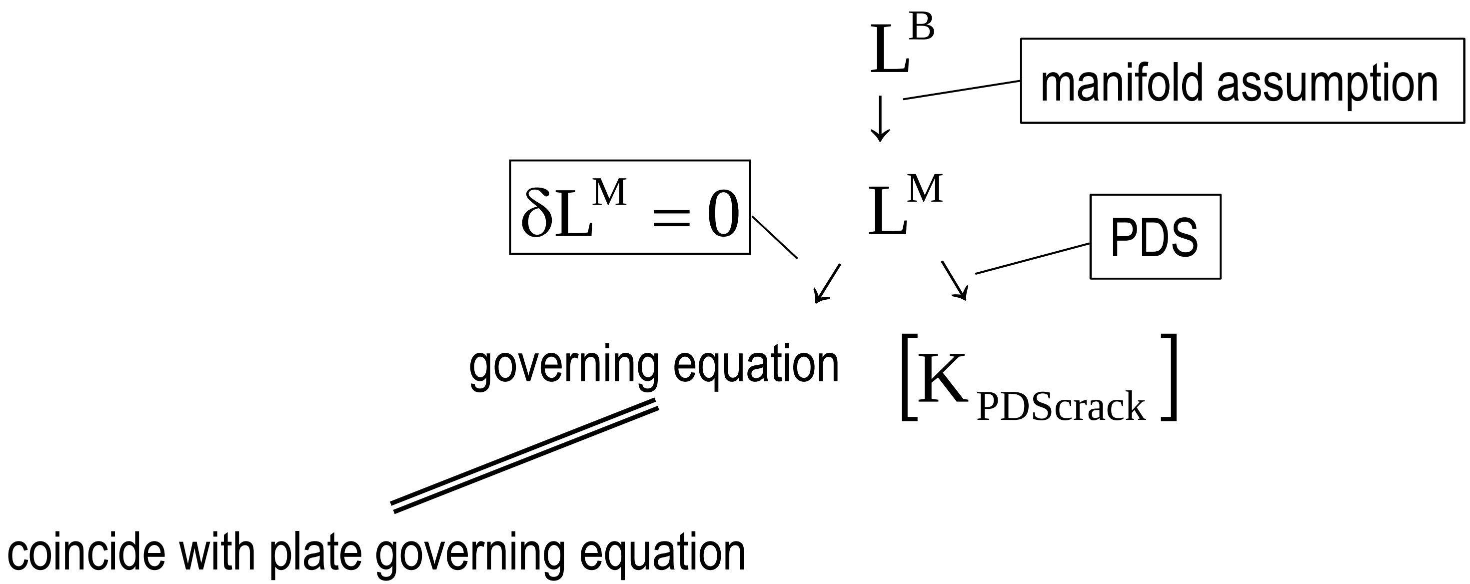
- Higher order polynomials included as if "expansion"

PDS could be set of *connected* Taylor expansion series

PLATE AS SIMPLE SHELL

- Examine usefulness of PDS with polynomials, using simpler structure element, i.e., plate element

- Development of cracked plate element

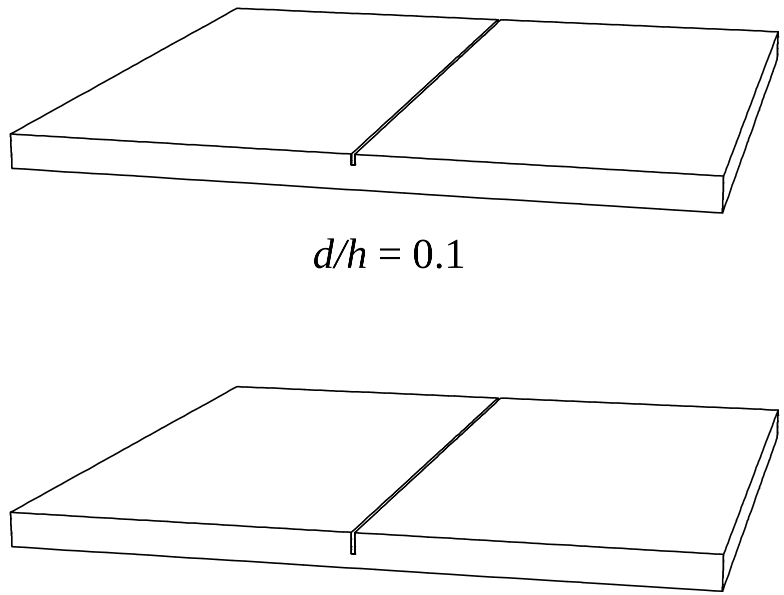


NUMERICAL EXPERIMENT

◆Problem setting

- $1.0 \times 1.0 \times 0.05$ m
- E = 200 GPa, ν = 0.3
- uniform pressure 100 N/mm^2
- clumped boundary condition
- crack with depth d at center

h = 0.05 m



d/h = 0.1

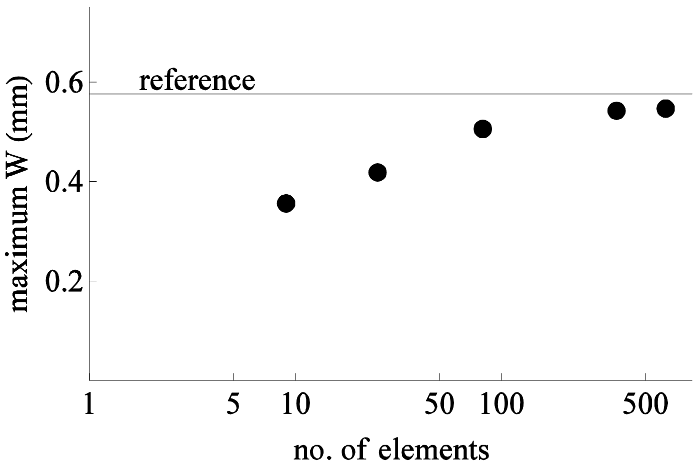
d/h = 0.7

plate

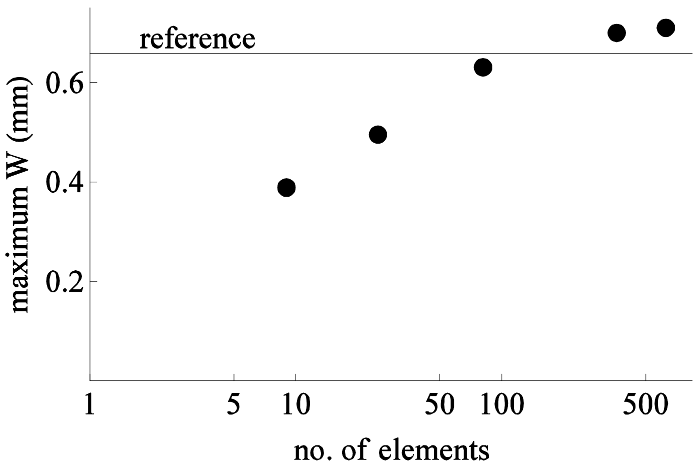
◆Mesh

- cracked plate element $25 \times 25 = 625$
- reference solid element 0,694 ～ 101,046

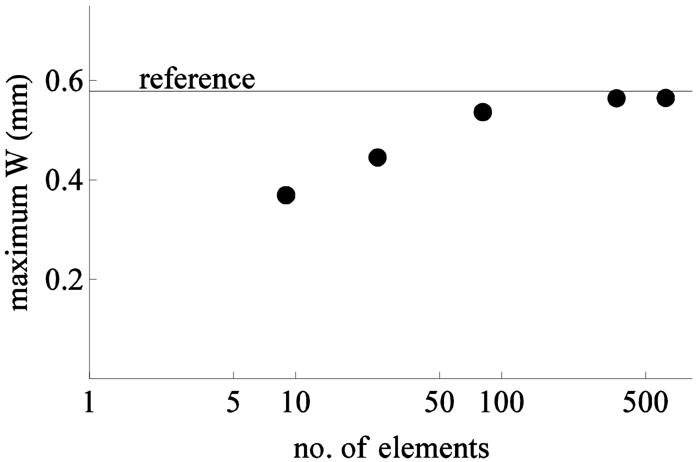
CONVERGENCE



d/*h* = 0.0

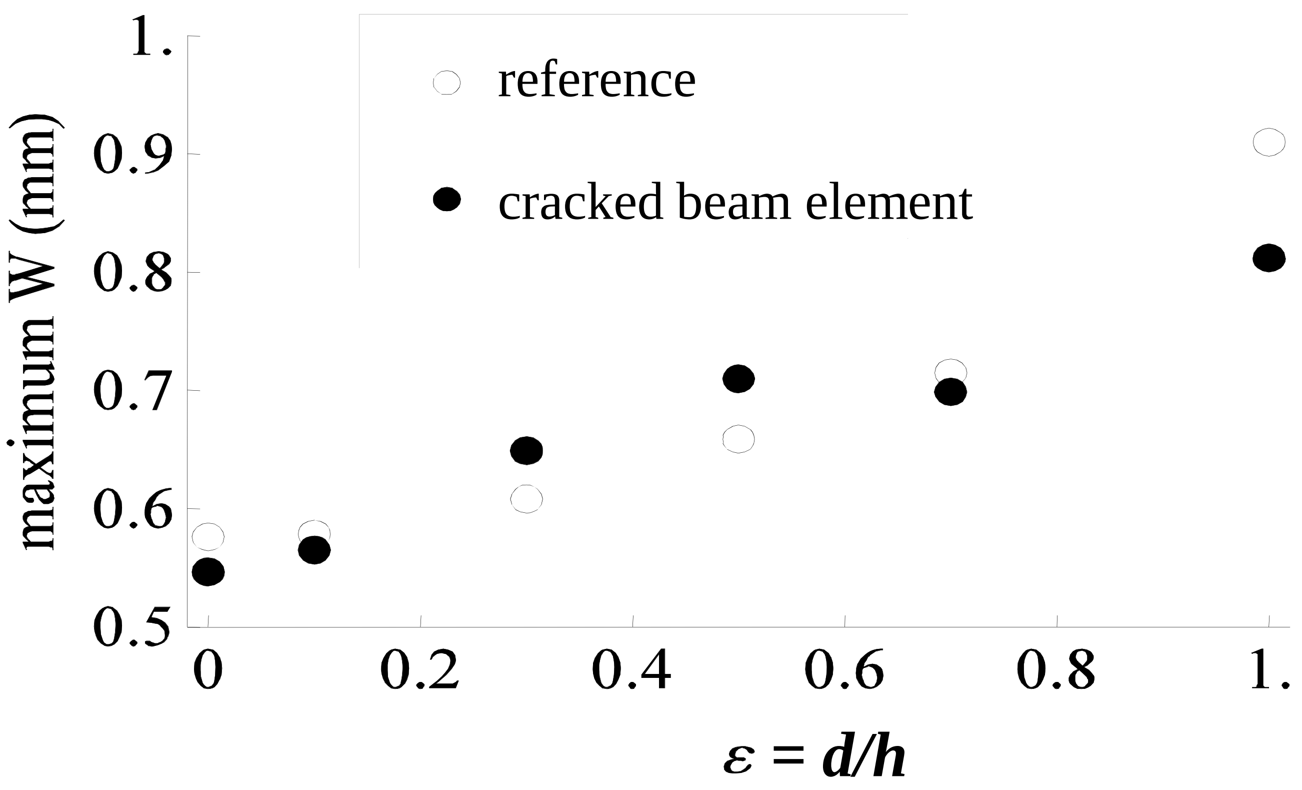


d/*h* = 0.5

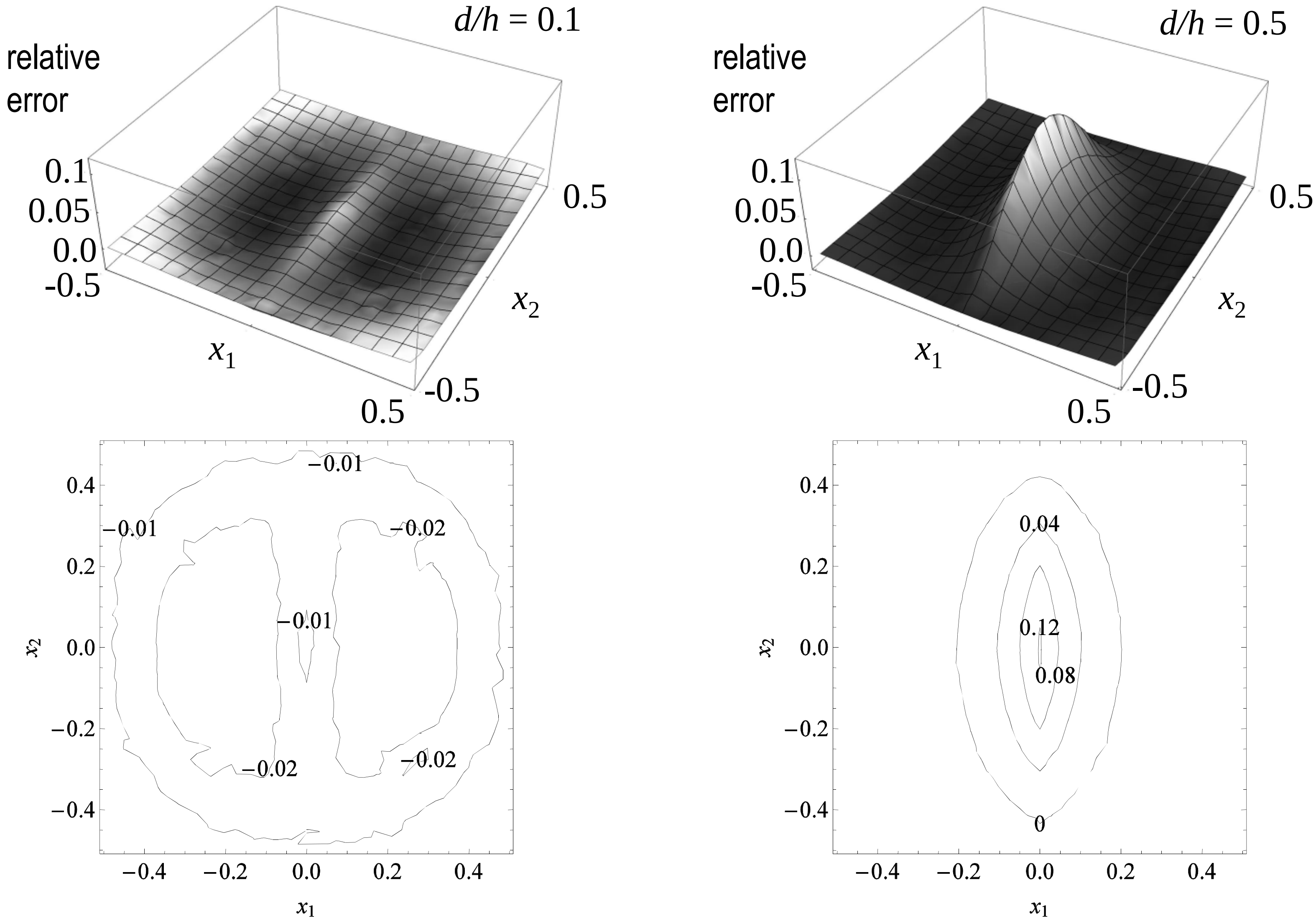


d/*h* = 0.1

DISPLACEMENT

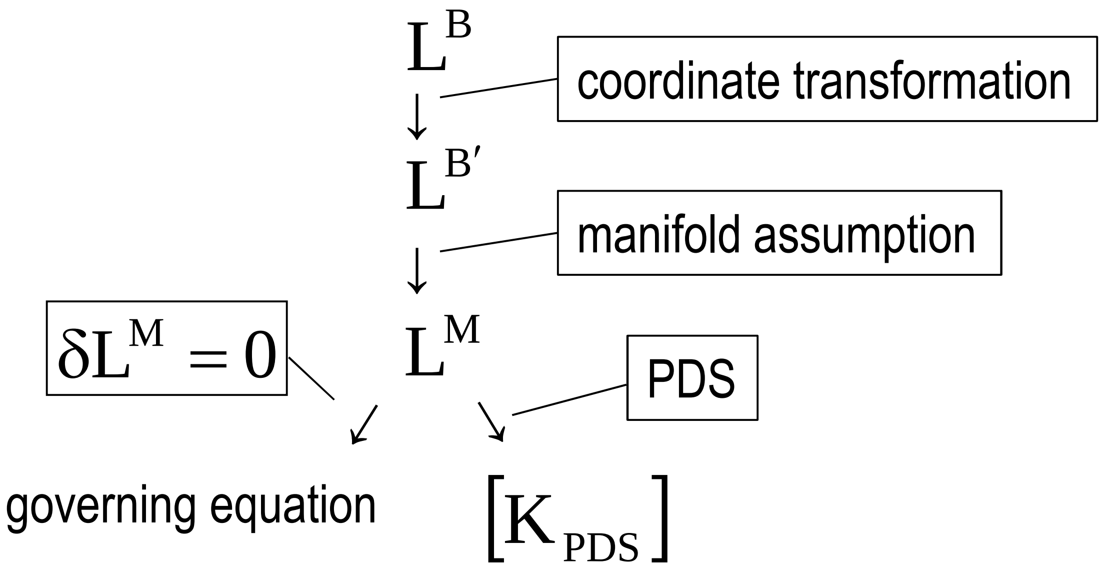


DISPLACEMENT



SHELL ELEMENT

- Discretize functions in L^M using PDS

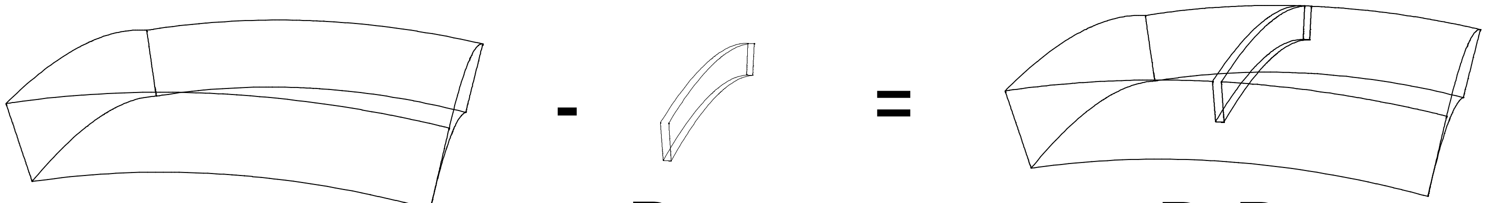


- Stress and strain is computed when $[K_{PDS}]$ is derived
  - 4 node 20 DOF (=$\{U, V, W, \theta_1, \theta_2\} \times 4$) element is derived

$[K_{PDS}] \neq [K_{FEM}]$: Need to verify performance

CRACKED SHELL

- Crack is modeled at boundary of Φ domains
  - Remove infinitesimal thin domain surrounding crack from Lagrangian



B $\quad$ B_{crack} $\quad$ $B - B_{crack}$

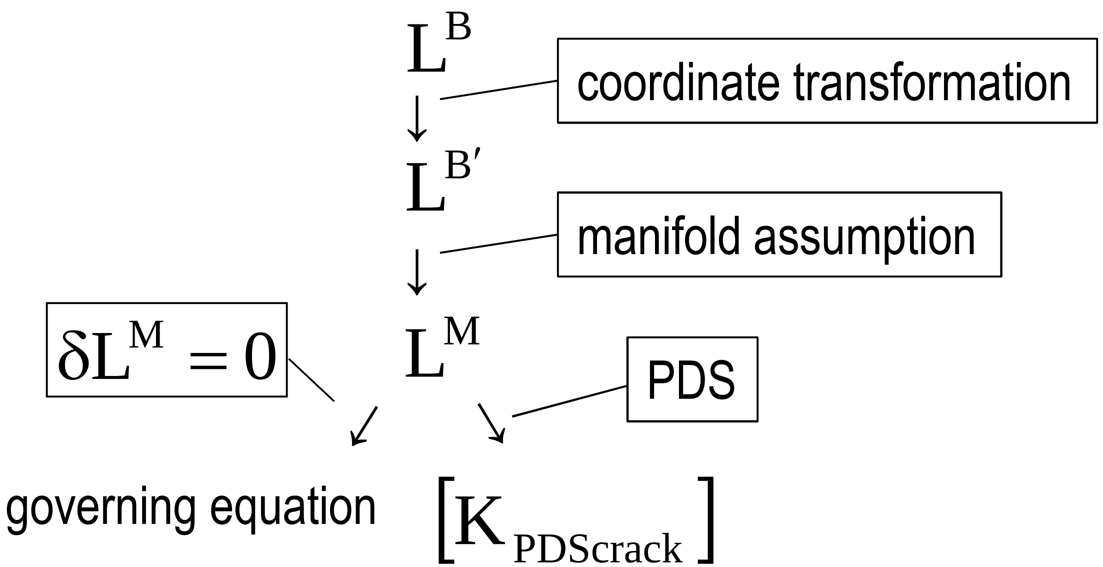
- Shell functional is recomputed

$$L^B(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_{B-B_{crack}} \frac{1}{2}\varepsilon_{ij} c_{ijkl} \varepsilon_{kl} + \sigma_{ij}(u_{i,j} - \varepsilon_{ij}) d\mathbf{x}$$

$$[K_{PDS}] \rightarrow [K_{PDScrack}]$$

SUMMARY

- Lagrangian of elastic solids + manifold + PDS

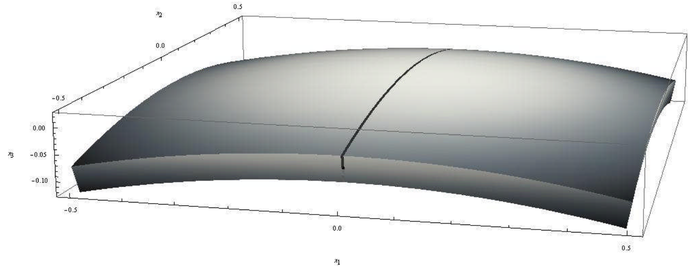


- Straightforward derivation, with some manipulation

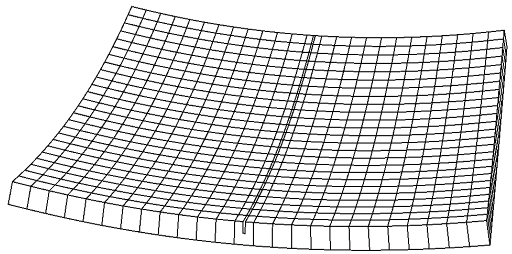
NUMERICAL EXPERIMENT

◆Problem Setting

- 1.0×1.0×0.05 m (κ = 0.4 /m)
- E = 200 GPa, ν = 0.3
- uniform pressure 100 N/mm^2
- clumped boundary condition
- crack with depth d at center



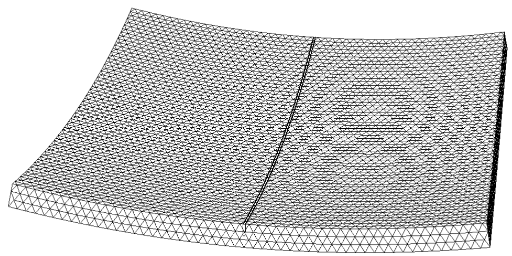
domain of analysis



cracked shell element

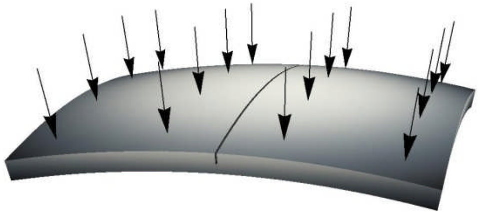
◆Mesh

- cracked shell element 25 × 25 = 625
- reference solid element 90,000 – 100,000



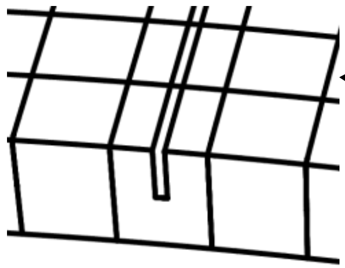
2nd order tetrahedron element

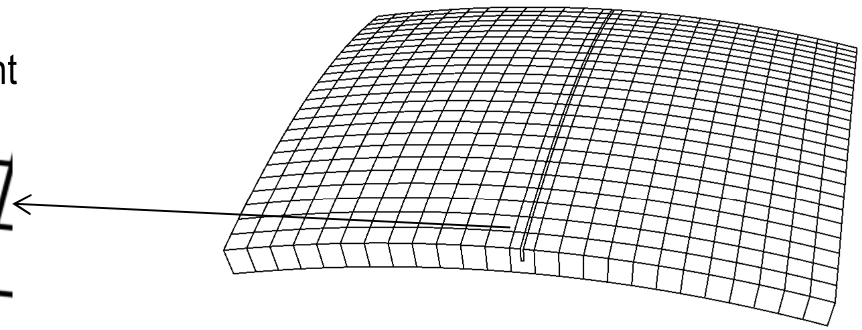
DETIAL OF CRACK MODELING



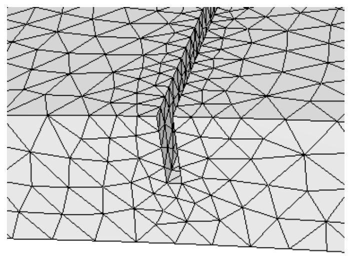
1.0 × 1.0 × 0.05 [m], Curvature: 0.4
E = 200 GPa, ν = 0.3
clamped edges
uniform load 1.0 MPa

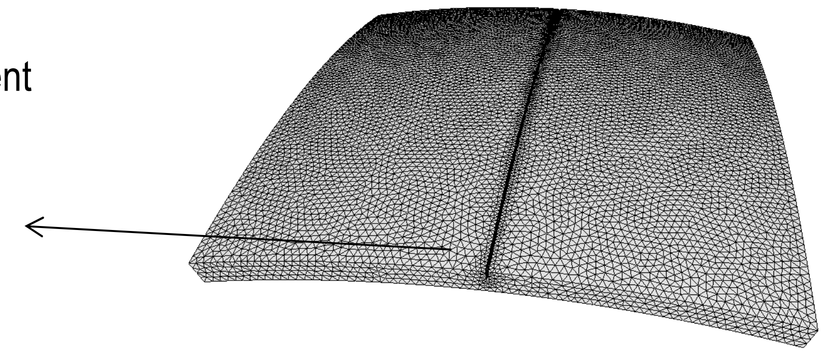
cracked shell element



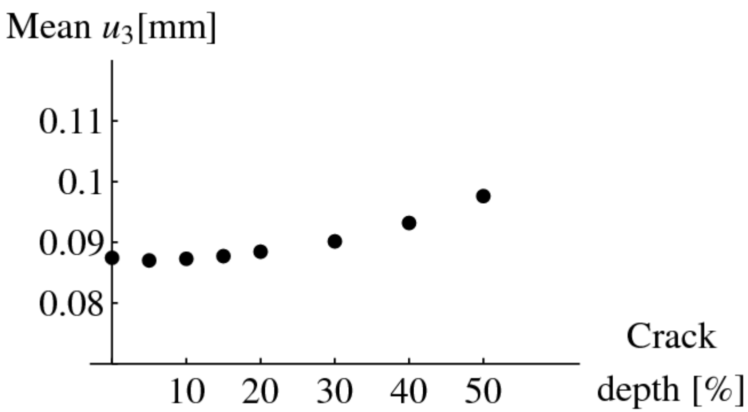


2nd order tetrahedron element

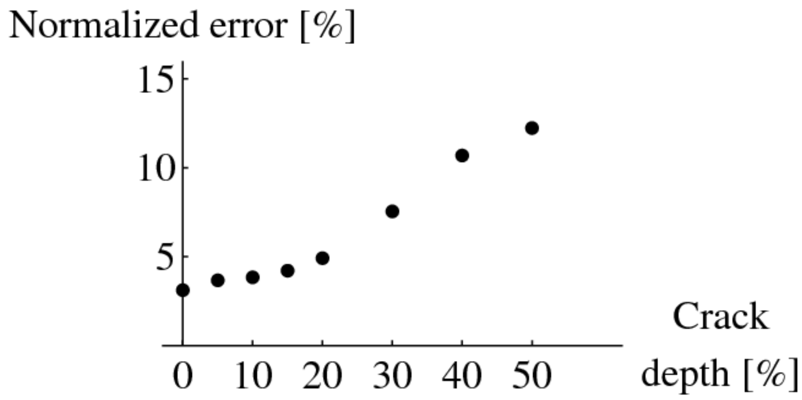




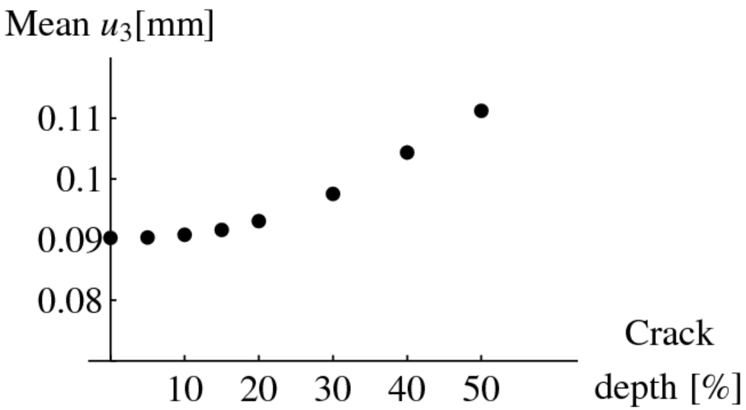
CONVERGENCE



reference



normalized error

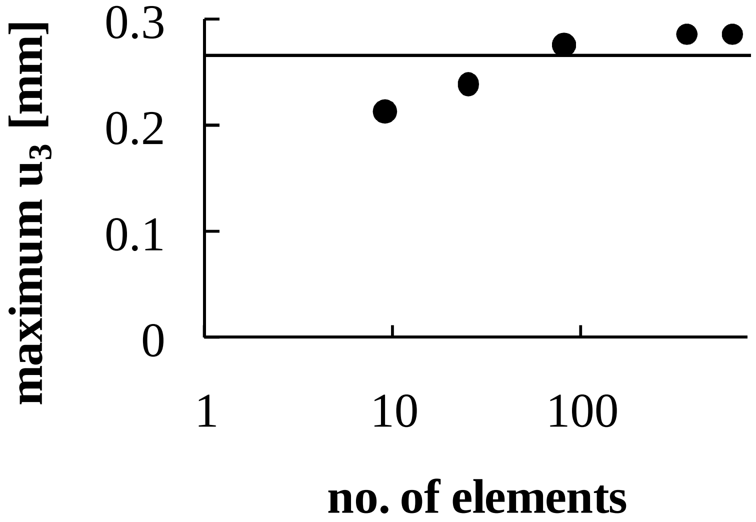


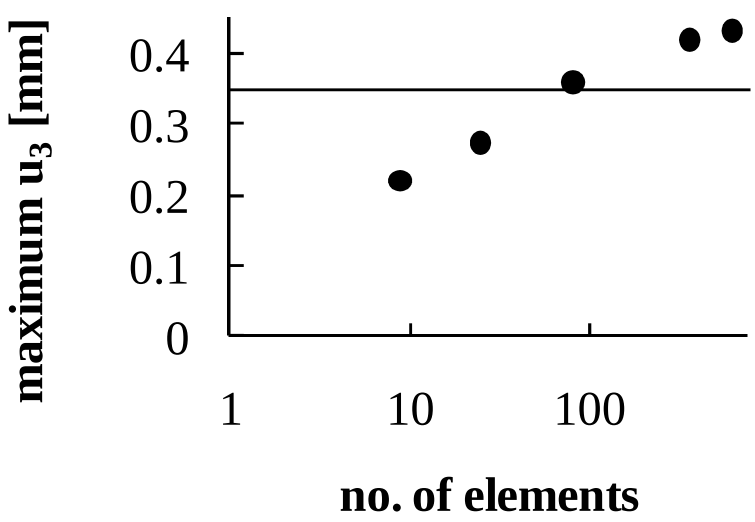
cracked shell element

Error is under 5% when crack is shallow.
Error exceeds 5% when crack is deep.

CONVERGENCE

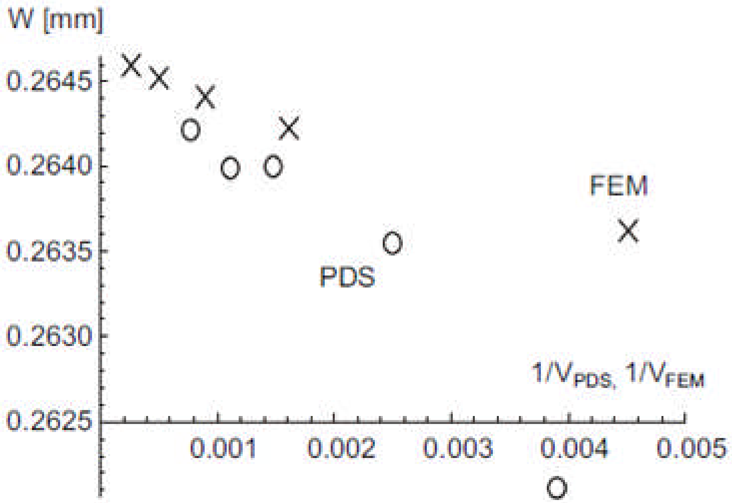
maximum displacement in thickness direction



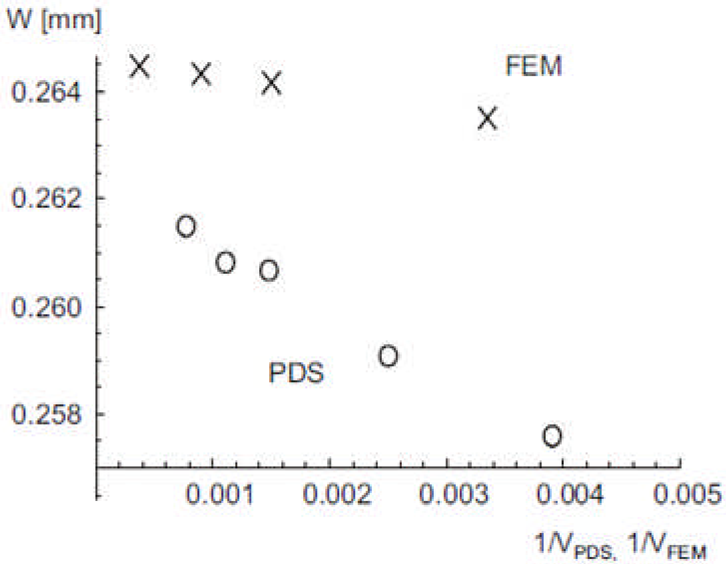


crack depth 10% 50%

CONVERGENCE



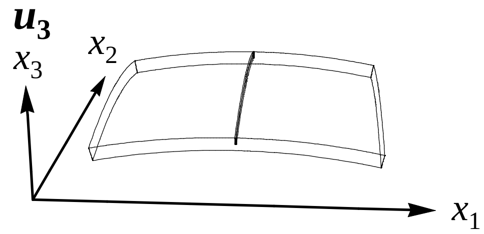
a) $d = 0.0$

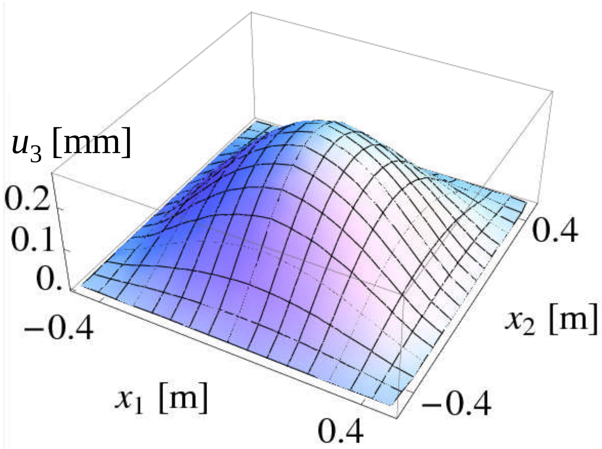


b) $d = 0.3$

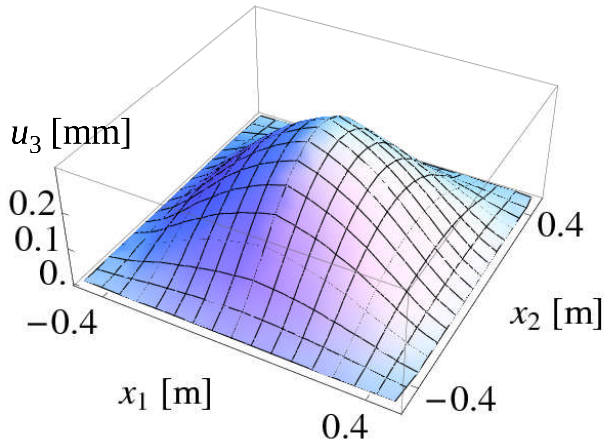
DISPLACEMENT

displacement in thickness direction
Crack depth is 20% of shell thickness

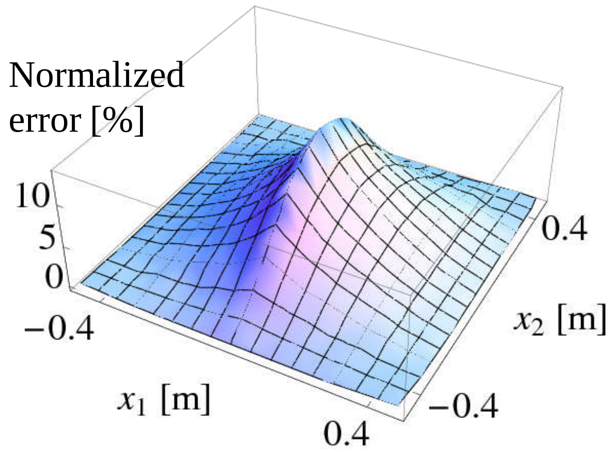




reference



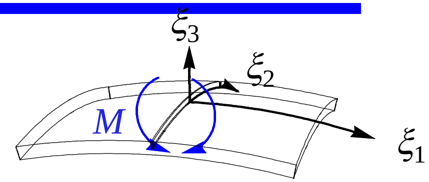
cracked shell element



normalized error

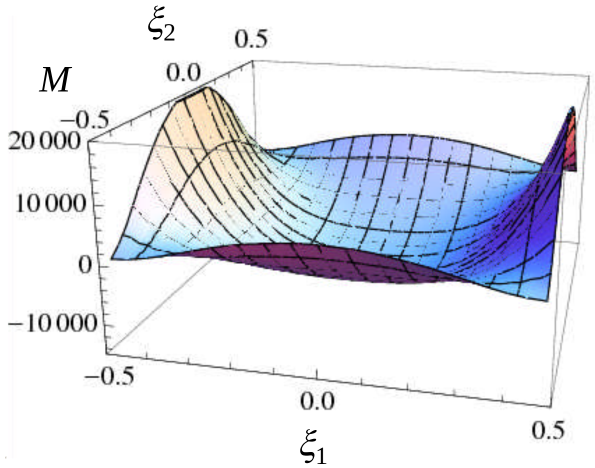
Error is about 12% at crack tip

MOMENT (NO CRACK)

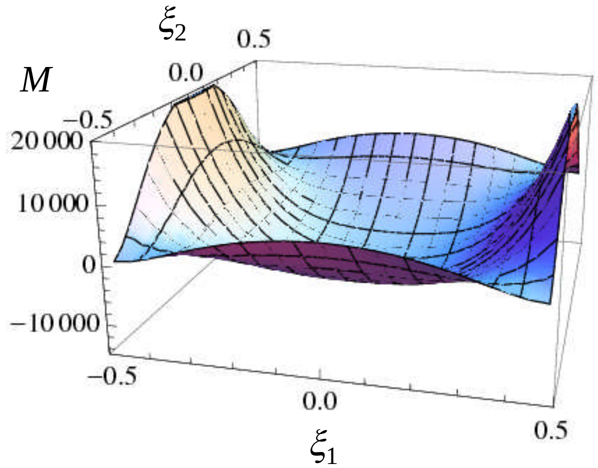


coordinates

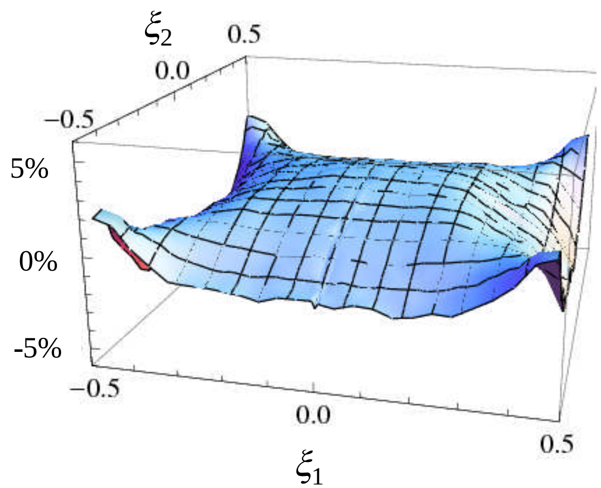
Moment $M = \int \sigma_1{}^1 \xi_3 \, d\xi_3$



reference



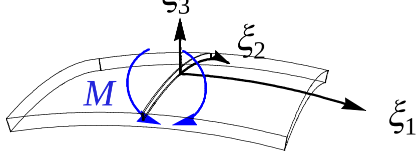
cracked shell element



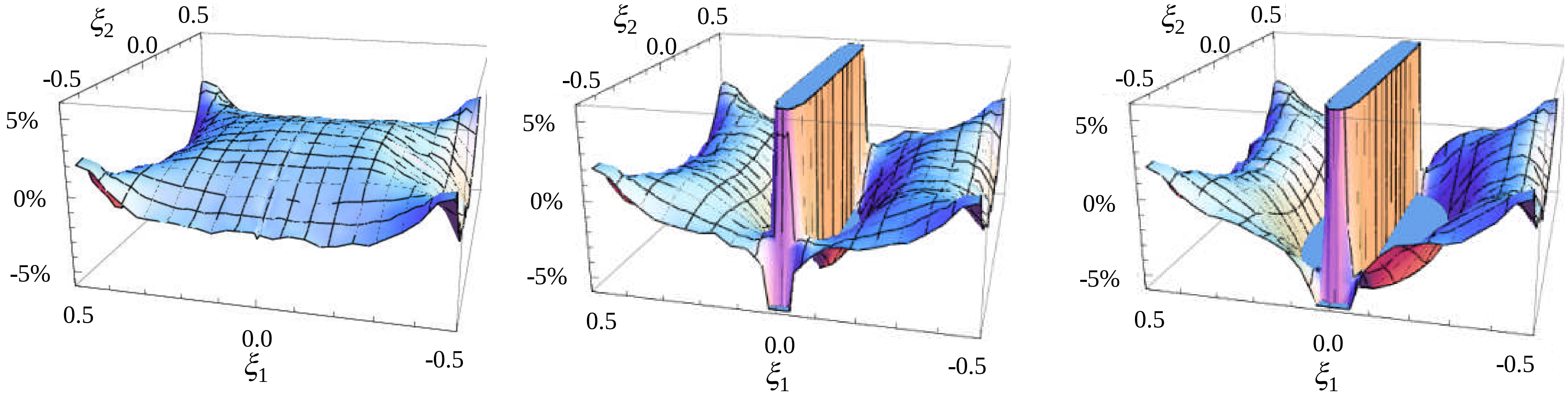
normalized error

Error is less than 2%

MOMENT



coordinates

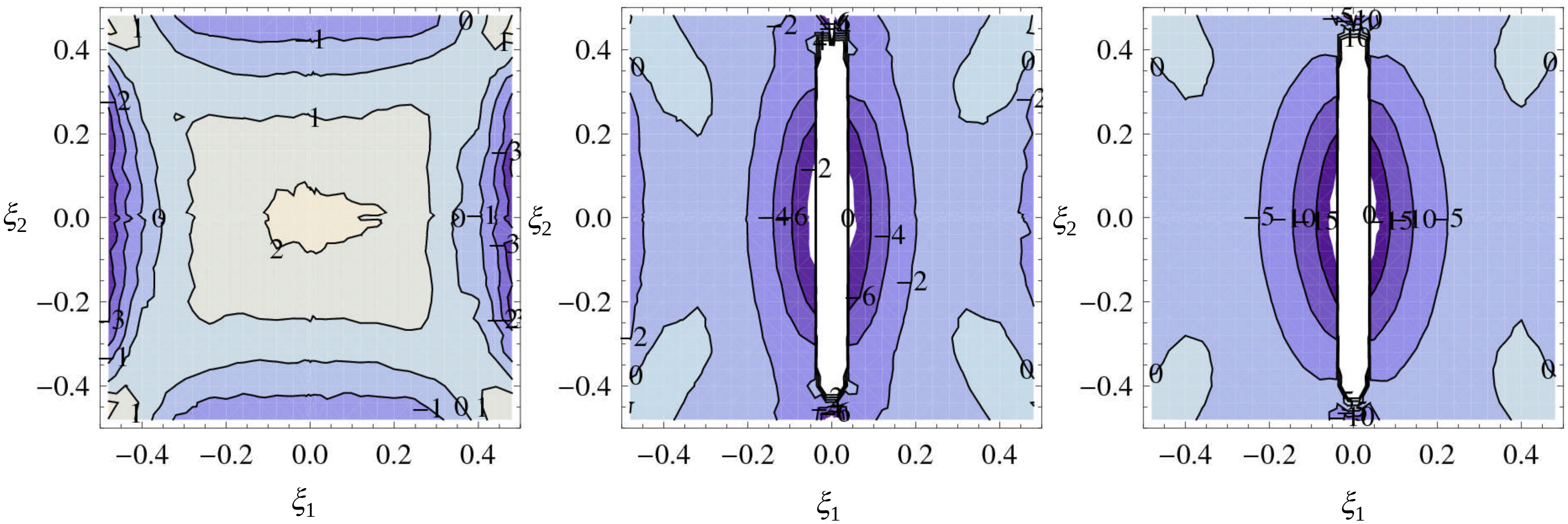


crack depth 0% 20% 30%

Error is under 5% when crack is shallow.
Error exceeds 5% when crack is deep.

Cracked shell element is valid for shallow cracks

MOMENT



crack depth 0% 20% 30%

Error is under 5% when crack is shallow.
Error exceeds 5% when crack is deep.

CONCLUDING REMARKS

- Cracked shell element is developed
  - applicable to crack of arbitrary depth
  - no need for node/element enrichment
  - limited accuracy

  not suitable for crack analysis, but could be used to evaluate structure performance

- Points in development
  - purely mathematical manipulation of 3D elastic continuum mechanics to structure mechanics
  - application of a new discretization scheme, PDS, for structure FEM element