Shape derivatives of cost functions in shape optimization problems

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§1 Introduction

Shape optimization problem of linear elastic body in $\mathbb{R}^d$, $d \in \{2, 3\}$

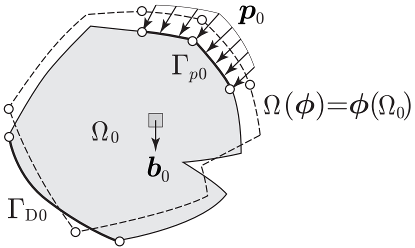


Fig. 1: Initial domain Ω_0 of linear elastic body and domain variation ϕ

§1 Introduction (cnt.)

Problem 1.1 (Shape optimization problem of linear elastic body)

For each $\Omega(\phi)$, let $\boldsymbol{u} : \Omega(\phi) \to \mathbb{R}^d$ satisfy

$$\begin{aligned} -\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{T}(\phi, \boldsymbol{u}) &= \boldsymbol{b}^{\mathrm{T}}(\phi) \quad \text{in } \Omega(\phi), \\ \boldsymbol{T}(\phi, \boldsymbol{u}) \boldsymbol{\nu} &= \boldsymbol{p}(\phi) \quad \text{on } \Gamma_p(\phi), \\ \boldsymbol{T}(\phi, \boldsymbol{u}) \boldsymbol{\nu} &= \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{N}}(\phi) \setminus \bar{\Gamma}_p(\phi), \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}}(\phi) \quad \text{on } \Gamma_{\mathrm{D}}(\phi), \end{aligned}$$

where $\boldsymbol{T}(\phi, \boldsymbol{u}) = \boldsymbol{C}(\phi) \boldsymbol{E}(\boldsymbol{u})$. Find $(\Omega(\phi), \boldsymbol{u})$ such that

$$\min_{(\boldsymbol{\phi}, \boldsymbol{u} \circ \boldsymbol{\phi}) \in \mathcal{O} \times U} \left\{ f^0(\Omega(\phi), \boldsymbol{u}) \mid f^1(\Omega(\phi)) \leq 0 \right\}.$$

where f^0 and f^1 are cost functions.

§1 Introduction (cnt.)

Using the solution $\boldsymbol{u}$, we call

$$f^0(\Omega(\boldsymbol{\phi}), \boldsymbol{u}) = -\int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} (\boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u})\boldsymbol{\nu}) \cdot \boldsymbol{u}_{\mathrm{D}} \, \mathrm{d}\gamma + \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{b} \cdot \boldsymbol{u} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}(\boldsymbol{\phi})} \boldsymbol{p} \cdot \boldsymbol{u} \, \mathrm{d}\gamma \tag{1.1}$$

the mean compliance. Moreover, we call

$$f^1(\Omega(\boldsymbol{\phi})) = \int_{\Omega(\boldsymbol{\phi})} \mathrm{d}x - m_0 \tag{1.2}$$

the constraint function for domain measure, where $m_0 > 0$ is a constant.

§1 Introduction (cnt.)

If we obtain the Fréchet derivatives with respect to domain variation φ as

$$f^{0\prime}\left(\Omega\left(\phi\right),\boldsymbol{u}\right)\left[\varphi\right]=\left\langle \boldsymbol{g}^{0},\varphi\right\rangle ,$$
$$f^{1\prime}\left(\Omega\left(\phi\right)\right)\left[\varphi\right]=\left\langle \boldsymbol{g}^{1},\varphi\right\rangle ,$$

we call $\boldsymbol{g}^0$ and $\boldsymbol{g}^1$ the shape derivatives of f^0 and f^1 respectively.

§1 Introduction (cnt.)

Moving boundary nodes of a finite element model in proportion to the shape derivative often meet with numerical instability such as oscillating shapes.

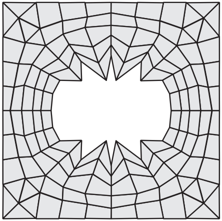


Fig. 2: Oscillating shape[BF84]

§1 Introduction (cnt.)

Gradient method in $H^1(\Omega(\phi);\mathbb{R}^d)$

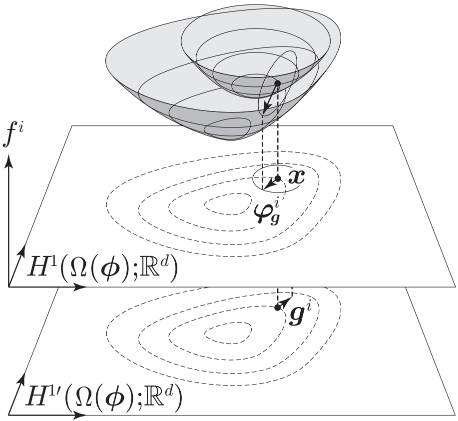


Fig. 3: Gradient method for shape optimization problem

§1 Introduction (cnt.)

Problem 1.2 (H^1 gradient method for shape optimization)

Let $a : H^1\left(\Omega\left(\phi\right);\mathbb{R}^d\right) \times H^1\left(\Omega\left(\phi\right);\mathbb{R}^d\right) \to \mathbb{R}$ be a coercive bilinear form such that there exists $\alpha > 0$ that satisfys

$$a\left(\boldsymbol{z}, \boldsymbol{z}\right) \geq \alpha \left\|\boldsymbol{z}\right\|^2$$

for all $\boldsymbol{z} \in H^1\left(\Omega\left(\phi\right);\mathbb{R}^d\right)$. For $\boldsymbol{g}^i\left(\boldsymbol{x}\right) \in H^{1\prime}\left(\Omega\left(\phi\right);\mathbb{R}^d\right)$, find $\boldsymbol{\varphi}_g^i \in H^1\left(\Omega\left(\phi\right);\mathbb{R}^d\right)$ such that

$$a\left(\boldsymbol{\varphi}_g^i, \boldsymbol{z}\right) = -\left\langle \boldsymbol{g}^i, \boldsymbol{z}\right\rangle.$$

for all $\boldsymbol{z} \in V$.

§1 Introduction (cnt.)

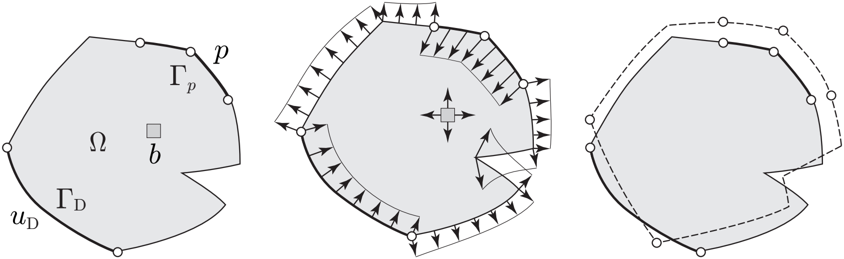
Problem 1.3 (Strong form of H^1 gradient method)

For $\boldsymbol{g}^i(\boldsymbol{x}) \in H^{1\prime}(\Omega(\phi); \mathbb{R}^d)$, find $\boldsymbol{y}_g^i \in H^1(\Omega(\phi); \mathbb{R}^d)$ such that

$$-\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{T}(\boldsymbol{\varphi}^i) + c\boldsymbol{\varphi}^{i\mathrm{T}} = \boldsymbol{g}_{\Omega}^{i\mathrm{T}} \quad \text{in } \Omega(\phi),$$

$$\boldsymbol{T}(\boldsymbol{\varphi}^i)\boldsymbol{\nu} = \boldsymbol{g}_{\Gamma}^i \quad \text{on } \partial\Omega(\phi).$$

§1 Introduction (cnt.)



Linear elastic problem

$(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{u}_D) \to \boldsymbol{u}$

Adjoint problem

$(\zeta^i_{\boldsymbol{u}}, \eta^i_{\boldsymbol{u}}) \mapsto \boldsymbol{v}^i$

H^1 gradient method

$\boldsymbol{g}^i \to \boldsymbol{\varphi}^i_g$

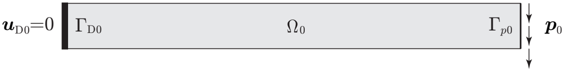
Reshaping

$\Omega \to \epsilon \boldsymbol{\varphi}^i_g(\Omega)$

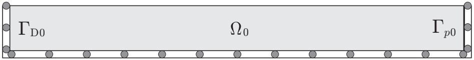
Fig. 4: H^1 gradient method for shape optimization

§1 Introduction (cnt.)

Numerical example



Linear elastic problem



Domain variation

Fig. 5: Boundary conditions for shape optimization problem of cantilever

§1 Introduction (cnt.)

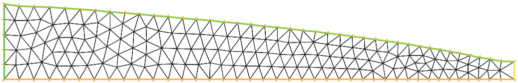


Fig. 6: Result of domain variation

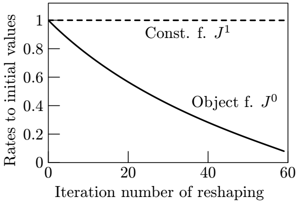


Fig. 7: Iteration history of cost functions

§1 Introduction (cnt.)

In the previous numerical examples, we assumed the followings.

1. Domain variation to the normal direction on $\Gamma_p \cup \Gamma_{\mathrm{D}}$ has been fixed.
2. $\boldsymbol{b}(\phi)$, $\boldsymbol{p}(\phi)$, $\boldsymbol{u}_{\mathrm{D}}(\phi)$, $\boldsymbol{C}(\phi)$ were assumed to be fixed in space, that is to be independent with respect to domain variation.

Then, we used the results of

$$\langle \boldsymbol{g}^0, \boldsymbol{\varphi} \rangle = \int_{\partial\Omega(\phi)\setminus(\Gamma_p\cup\Gamma_{\mathrm{D}})} \left(-\boldsymbol{T}(\phi, \boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{u}) + 2\boldsymbol{b}(\phi) \cdot \boldsymbol{u}\right) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma,$$

$$\langle \boldsymbol{g}^1, \boldsymbol{\varphi} \rangle = \int_{\partial\Omega(\phi)\setminus(\Gamma_p\cup\Gamma_{\mathrm{D}})} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma.$$

§1 Introduction (cnt.)

In the early stage of our study, we presented numerical examples moving Γ_p[AW96].

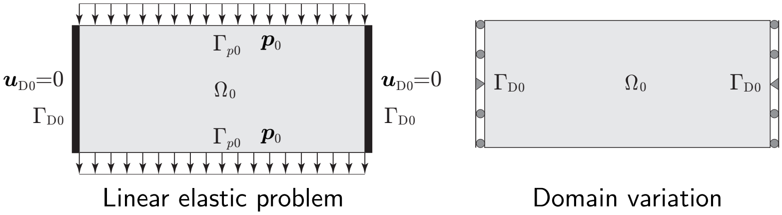


Fig. 8: Boundary conditions for shape optimization problem of plate both sides fixed

§1 Introduction (cnt.)

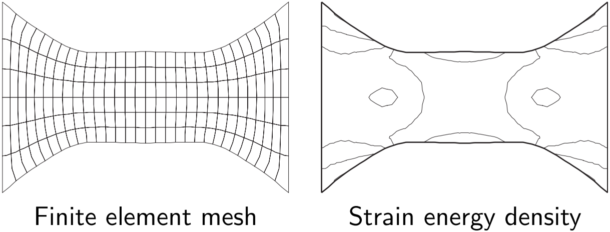
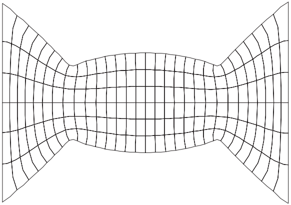
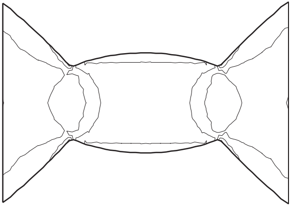


Fig. 9: Result of domain variation assuming $\boldsymbol{p}$ is fixed with space or fixed with material

§1 Introduction (cnt.)



Finite element mesh



Strain energy density

Fig. 10: Result of domain variation assuming $\boldsymbol{p}$ is of varying with boundary measure

§1 Introduction (cnt.)

Goal

1. To define variation rules with respect to domain variation of functions such as $\boldsymbol{b}(\boldsymbol{\phi})$, $\boldsymbol{p}(\boldsymbol{\phi})$, $\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi})$, $\boldsymbol{C}(\boldsymbol{\phi})$.
2. To show furmulae required for computing the shape derivatives of functions applying the various variation rules.
3. To demonstrate the results for shape derivatives of mean compliance in shape optimization problem of linear elastic body.

§2 Set of mappings for domain variations

Let us define a set of mappings $\phi : \Omega_0 \to \mathbb{R}^d$ for domain variations.

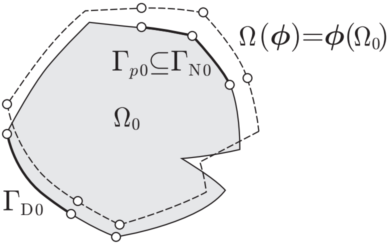


Fig. 11: Initial domain Ω_0 of $W^{1,\infty}$ class and mapping ϕ for domain variation

§2 Set of mappings for domain variations (cnt.)

We set an initial domain $\Omega_0 \subset \mathbb{R}^d$ as follows. Let C^1_{pw} denote the piecewise C^1.

1. Let $\partial\Omega_0$ be the Lipschitz ($W^{1,\infty}$ class) domain.
2. Let $\Gamma_{\mathrm{D}\,0} \subset \partial\Omega_0$, $|\Gamma_{\mathrm{D}\,0}| > 0$, be the Dirichlet boudary of C^1_{pw} class.
3. Let $\Gamma_{\mathrm{N}\,0} = \partial\Omega_0 \setminus \bar{\Gamma}_{\mathrm{D}\,0}$ be the Neumann boundary, and $\Gamma_{p\,0} \subset \Gamma_{\mathrm{N}\,0}$ be the nonhomogeneous Neumann boundary of C^1_{pw} class.
4. Let $\Gamma^i_{\eta\,0} \subseteq \partial\Omega_0$ be the boundary of C^1_{pw} class on which non-zero integrants of boundary integrals in cost functions f^i are defined. Moreover, let $\Gamma_{\eta\,0} = \bigcup_{i=0}^{m} \Gamma^i_{\eta\,0}$.

§2 Set of mappings for domain variations (cnt.)

Let us define a set for domain variations ϕ. Hereafter, we denote the Jacobi matrix and Jacobian of $\phi \in W^{1,\infty}\left(\Omega_0;\mathbb{R}^d\right)$ at $\boldsymbol{x} \in \Omega_0$ by

$$\boldsymbol{F}_0\left(\boldsymbol{\phi}\right) = \boldsymbol{\phi}_{\boldsymbol{x}^{\mathrm{T}}} = \left(\frac{\partial \phi_i}{\partial x_j}\right) \in L^{\infty}\left(\Omega_0;\mathbb{R}^{d\times d}\right), \tag{2.1}$$

$$\omega_0\left(\boldsymbol{\phi}\right) = \det \boldsymbol{F}_0\left(\boldsymbol{\phi}\right) \in L^{\infty}\left(\Omega_0;\mathbb{R}\right). \tag{2.2}$$

Let

$$V = \left\{\boldsymbol{\phi} \in W^{1,\infty}\left(\Omega_0;\mathbb{R}^d\right) \;\middle|\; \operatorname*{ess\,inf}_{\boldsymbol{x}\in\Omega_0} \omega_0\left(\boldsymbol{\phi}\right) > 0 \tag{2.3}$$

$$\boldsymbol{\phi} : C^1_{\mathrm{pw}} \text{ class on } \Gamma_{\eta\,0} \cup \Gamma_{p\,0} \cup \Gamma_{\mathrm{D}\,0}\right\} \tag{2.4}$$

be the real Banach space for domain variation.

§2 Set of mappings for domain variations (cnt.)

Here, we use the following result[Kim08].

Proposition 2.1 (Sufficient condition for one-to-one mapping)

If $\phi \in W^{1,\infty}\left(\Omega_0; \mathbb{R}^d\right)$ satisfies

$$\|\phi - \phi_0\|_{W^{1,\infty}(\Omega_0;\ \mathbb{R}^d)} < 1, \tag{2.5}$$

ϕ is injective (one-to-one) function form Ω_0 to an open set $\phi\left(\Omega_0\right)$, and ϕ and ϕ^{-1} are uniform Lipschitz continuous.

Using the result, we define a set of domain variation by

$$\mathcal{O} = \left\{ \phi \in V \;\middle|\; \|\phi - \phi_0\|_{W^{1,\infty}(\Omega_0;\ \mathbb{R}^d)} < 1 \right\}. \tag{2.6}$$

§2 Set of mappings for domain variations (cnt.)

Moreover, fixing $\phi \in \mathcal{O}$, let

$$V(\phi) = \left\{ \varphi \in W^{1,\infty}\left(\Omega(\phi); \mathbb{R}^d\right) \;\middle|\; \operatorname*{ess\,inf}_{x \in \Omega(\phi)} \omega(\varphi) > 0 \right\} \qquad (2.7)$$

be the real Banach space for domain variaition from $\Omega(\phi)$, where

$$F(\varphi) = \varphi_{\phi^{\mathrm{T}}} = \left(\frac{\partial \varphi_i}{\partial \phi_j} \right) \in L^{\infty}\left(\Omega(\phi); \mathbb{R}^{d \times d}\right), \qquad (2.8)$$

$$\omega(\varphi) = \det F(\varphi) \in L^{\infty}\left(\Omega(\phi); \mathbb{R}\right) \qquad (2.9)$$

Jacobi matrix and Jacobian of $\varphi \in V(\phi)$ at $x \in \Omega(\phi)$.

§2 Set of mappings for domain variations (cnt.)

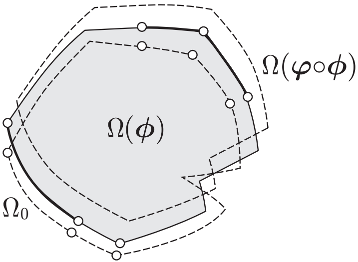


Fig. 12: Domain variation by $\boldsymbol{\varphi}$

§2 Set of mappings for domain variations (cnt.)

Using the definitions above, denoting the domain varied by $\varphi \in V(\phi)$ as $\Omega(\varphi \circ \phi) = (\varphi \circ \phi)(\Omega_0)$, and following the definition of (2.1) and (2.2),

$$\boldsymbol{F}_0(\varphi \circ \phi) = \boldsymbol{F}(\varphi)\,\boldsymbol{F}_0(\phi)$$
$$\omega_0(\varphi \circ \phi) = \omega(\varphi)\,\omega_0(\phi)$$

hold.

§3 Variation rule of function

Let us define variation rule of functions. Let $\phi \in \mathcal{O}$ and $D \supset \Omega_0$ be a fixed large domain.

1. If $\boldsymbol{x} \in \Omega_0$ moves to $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x})$ and $v : D \to \mathbb{R}$ varies as

$$\hat{v}(\boldsymbol{z}) = v \circ \boldsymbol{\phi}(\boldsymbol{x}),$$

   we call $\hat{v} : \Omega(\phi) \to \mathbb{R}$ the function fixed with space.

2. If $\boldsymbol{x} \in \Omega_0$ moves to $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x})$ and $v : \Omega_0 \to \mathbb{R}$ varies as

$$\hat{v}(\boldsymbol{z}) = v \circ \boldsymbol{\phi}^{-1}(\boldsymbol{z}) = v(\boldsymbol{x}),$$

   we call $\hat{v} : \Omega(\phi) \to \mathbb{R}$ the function fixed with material (Fig. 13).

§3 Variation rule of function (cnt.)

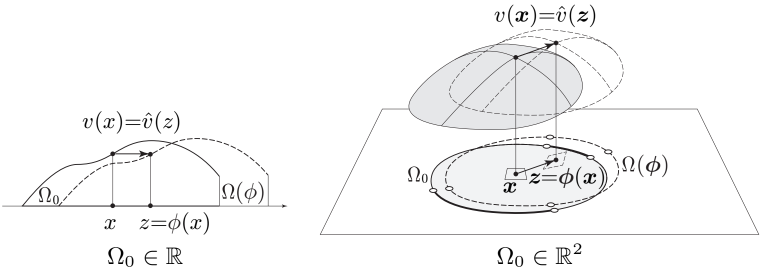


Fig. 13: Function fixed with material

§3 Variation rule of function (cnt.)

3. Let $\mathrm{d}x$ be a small measure of Ω_0 (small volume at $d = 3$), $\mathrm{d}z$ be the small measure of $\Omega(\boldsymbol{\phi})$ corresponding to $\mathrm{d}x$. If $\boldsymbol{x} \in \Omega_0$ moves to $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x})$ and $v : \Omega_0 \to \mathbb{R}$ varies as

$$\hat{v}(\boldsymbol{z})\,\mathrm{d}z = v(\boldsymbol{x})\,\mathrm{d}x,$$

we call $\hat{v} : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ the function varying with domain measure. Moreover, let $\mathrm{d}\gamma$ be a small measure of $\partial\Omega_0$, $\mathrm{d}\gamma^{\boldsymbol{\phi}}$ be the small measure of $\partial\Omega(\boldsymbol{\phi})$ corresponding to $\mathrm{d}\gamma$. If $\boldsymbol{x} \in \partial\Omega_0$ moves to $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x})$ and $v : \partial\Omega_0 \to \mathbb{R}$ varies as

$$\hat{v}(\boldsymbol{z})\,\mathrm{d}\gamma^{\boldsymbol{\phi}} = v(\boldsymbol{x})\,\mathrm{d}\gamma,$$

we call $\hat{v} : \partial\Omega(\boldsymbol{\phi}) \to \mathbb{R}$ the function varying with boundary measure.

§4 Shape optimization problem of linear elastic body

Let us demonstrate the results for shape derivatives of mean compliance in shape optimization problem of linear elastic body.

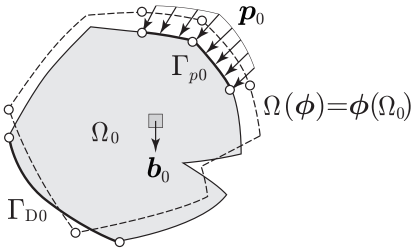


Fig. 14: Initial domain Ω_0 of linear elastic body and domain variation ϕ

§4.1 Linear elastic problem

We set $U = H^1\left(\Omega_0; \mathbb{R}^d\right)$.

Problem 4.1 (Linear elastic problem)

Let $\phi \in \mathcal{O}$, $\boldsymbol{b}_0$, $\boldsymbol{p}_0$, $\boldsymbol{u}_{\mathrm{D}\,0} : \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{C} : \mathbb{R}^d \to \mathbb{R}^{d\times d\times d\times d}$ be given, and $\boldsymbol{b}(\phi)$, $\boldsymbol{p}(\phi)$, $\boldsymbol{u}_{\mathrm{D}}(\phi)$, $\boldsymbol{C}(\phi)$ vary from $\boldsymbol{b}_0$, $\boldsymbol{p}_0$, $\boldsymbol{u}_{\mathrm{D}\,0}$, $\boldsymbol{C}_0$ by assigned variation rules. Find $\boldsymbol{u} \circ \phi \in U$ such that

$$
\begin{aligned}
-\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{T}(\phi, \boldsymbol{u}) &= \boldsymbol{b}^{\mathrm{T}}(\phi) \quad \text{in } \Omega(\phi), \\
\boldsymbol{T}(\phi, \boldsymbol{u})\,\boldsymbol{\nu} &= \boldsymbol{p}(\phi) \quad \text{on } \Gamma_p(\phi), \\
\boldsymbol{T}(\phi, \boldsymbol{u})\,\boldsymbol{\nu} &= \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{N}}(\phi) \setminus \bar{\Gamma}_p(\phi), \\
\boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}}(\phi) \quad \text{on } \Gamma_{\mathrm{D}}(\phi),
\end{aligned}
$$

where $\boldsymbol{T}(\phi, \boldsymbol{u}) = \boldsymbol{C}(\phi)\,\boldsymbol{E}(\boldsymbol{u})$.

§4.2 Shape derivative of mean compliance

Using the solution of Problem 4.1 $\boldsymbol{u} = \boldsymbol{v}^0$, the shape derivative of f^0 is obtained as the form of

$$f^{0\prime}(\Omega(\phi), \boldsymbol{u})[\boldsymbol{\varphi}] = \left\langle \boldsymbol{g}^0_{\Omega\, T} + \boldsymbol{g}^0_{\Omega\, b}, \boldsymbol{\varphi} \right\rangle_\Omega + \left\langle \boldsymbol{g}^0_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} + \left\langle \boldsymbol{g}^0_{\Gamma_\mathrm{D}}, \boldsymbol{\varphi} \right\rangle_{\Gamma_\mathrm{D}}. \tag{4.1}$$

These forms depend on the variation rule of $\boldsymbol{b}(\phi)$, $\boldsymbol{p}(\phi)$, $\boldsymbol{u}_\mathrm{D}(\phi)$, $\boldsymbol{C}(\phi)$. We show the results for the forms below.

In the followings, the solution $\boldsymbol{u} = \boldsymbol{v}^0$ of Problem 4.1 is assumed as fixing in space because of satisfying the stationaly condition of the Lagrange funcion of the shape optimization problem.

We use the notation of $\partial_\varphi = \boldsymbol{\varphi} \cdot \boldsymbol{\nabla}$.

§4.2 Shape derivative of mean compliance (cnt.)

1. If $\boldsymbol{C}(\boldsymbol{\phi})$ and $\boldsymbol{b}(\boldsymbol{\phi})$ are fixing in space, we have

$$\left\langle \boldsymbol{g}^0_{\Omega\, T}, \boldsymbol{\varphi} \right\rangle_\Omega = \int_{\Omega(\boldsymbol{\phi})} \Big[-\partial_{\varphi} \left\{ \boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \cdot \boldsymbol{E}\left(\boldsymbol{v}^0\right) \right\} - \left\{ \boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \cdot \boldsymbol{E}\left(\boldsymbol{v}^0\right) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big] \mathrm{d}x, \tag{4.2}$$

$$\left\langle \boldsymbol{g}^0_{\Omega\, b}, \boldsymbol{\varphi} \right\rangle_\Omega = \int_{\Omega(\boldsymbol{\phi})} \Big[\partial_{\varphi} \left\{ \boldsymbol{b} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \right\} + \left\{ \boldsymbol{b} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big] \mathrm{d}x. \tag{4.3}$$

Moreover, if $\partial\Omega(\boldsymbol{\phi})$ is C^1_{pw} class, we have

$$\left\langle \boldsymbol{g}^0_{\Omega\, T}, \boldsymbol{\varphi} \right\rangle_\Omega = \int_{\partial\Omega(\boldsymbol{\phi})} - \left\{ \boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \cdot \boldsymbol{E}\left(\boldsymbol{v}^0\right) \right\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma = \left\langle \boldsymbol{g}^0_{\partial\Omega\, T}, \boldsymbol{\varphi} \right\rangle_{\partial\Omega}, \tag{4.4}$$

$$\left\langle \boldsymbol{g}^0_{\Omega\, b}, \boldsymbol{\varphi} \right\rangle_\Omega = \int_{\partial\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{b} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \right\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma = \left\langle \boldsymbol{g}^0_{\partial\Omega\, b}, \boldsymbol{\varphi} \right\rangle_{\partial\Omega}. \tag{4.5}$$

§4.2 Shape derivative of mean compliance (cnt.)

If $\boldsymbol{p}(\boldsymbol{\phi})$, $\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi})$ and $\boldsymbol{C}(\boldsymbol{\phi})$ are fixed in space, recalling that $\Gamma_p(\boldsymbol{\phi}) \cup \Gamma_{\mathrm{D}}(\boldsymbol{\phi})$ is C^1_{pw} class with $\boldsymbol{\phi} \in \mathcal{O}$, we have

$$\left\langle \boldsymbol{g}^0_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} = \int_{\Gamma_p(\boldsymbol{\phi})} \left[\partial_{\boldsymbol{\varphi}} \left\{ \boldsymbol{p} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \right\} + \boldsymbol{p} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu} \right) \right\} \right] \mathrm{d}\gamma, \tag{4.6}$$

$$\begin{aligned} \langle \boldsymbol{g}^0_{\Gamma_{\mathrm{D}}}, \boldsymbol{\varphi} \rangle_{\Gamma_{\mathrm{D}}} = \int_{\Gamma_{\mathrm{D}}} \Big[& \partial_{\boldsymbol{\varphi}} \Big\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left(\boldsymbol{T}\left(\boldsymbol{\phi}, \boldsymbol{v}^0\right) \boldsymbol{\nu} \right) \\ & + \left(\boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \boldsymbol{\nu} \right) \cdot \left(\boldsymbol{v}^0 - \boldsymbol{u}_{\mathrm{D}} \right) \Big\} \\ & + \left\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left(\boldsymbol{T}\left(\boldsymbol{\phi}, \boldsymbol{v}^0\right) \boldsymbol{\nu} \right) + \left(\boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \boldsymbol{\nu} \right) \cdot \left(\boldsymbol{v}^0 - \boldsymbol{u}_{\mathrm{D}} \right) \right\} \\ & \times \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu} \right) \right\} \Big] \mathrm{d}\gamma. \end{aligned} \tag{4.7}$$

§4.2 Shape derivative of mean compliance (cnt.)

Moreover, if $\Gamma_p(\phi) \cup \Gamma_{\mathrm{D}}(\phi)$ is C^2_{pw} class, we have

$$\left\langle \boldsymbol{g}^0_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} = \int_{\Gamma_p(\phi)} (\partial_\nu + \kappa) \left\{ \boldsymbol{p} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \right\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\partial\Gamma_p(\phi) \cup \Theta} \boldsymbol{p} \cdot \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}a, \tag{4.8}$$

$$\begin{aligned} \left\langle \boldsymbol{g}^0_{\Gamma_{\mathrm{D}}}, \boldsymbol{\varphi} \right\rangle_{\Gamma_{\mathrm{D}}} = & \int_{\Gamma_{\mathrm{D}}(\phi)} (\partial_\nu + \kappa) \Big\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left(\boldsymbol{T}\left(\phi, \boldsymbol{v}^0\right) \boldsymbol{\nu} \right) \\ & + \left(\boldsymbol{T}(\phi, \boldsymbol{u}) \boldsymbol{\nu} \right) \cdot \left(\boldsymbol{v}^0 - \boldsymbol{u}_{\mathrm{D}} \right) \Big\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\ & + \int_{\partial\Gamma_{\mathrm{D}}(\phi) \cup \Theta} \Big\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left(\boldsymbol{T}\left(\phi, \boldsymbol{v}^0\right) \boldsymbol{\nu} \right) \\ & + \left(\boldsymbol{T}(\phi, \boldsymbol{u}) \boldsymbol{\nu} \right) \cdot \left(\boldsymbol{v}^0 - \boldsymbol{u}_{\mathrm{D}} \right) \Big\} \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}a. \end{aligned} \tag{4.9}$$

§4.2 Shape derivative of mean compliance (cnt.)

2. If $\boldsymbol{C}(\phi)$ and $\boldsymbol{b}(\phi)$ are fixed with material, considering that $\boldsymbol{u} = \boldsymbol{v}^0$ is fixed in space, we have

$$\langle \boldsymbol{g}^0_{\Omega\, T}, \boldsymbol{\varphi} \rangle_\Omega = \int_{\Omega(\phi)} \Big[-\partial_\varphi \boldsymbol{E}(\boldsymbol{u}) \cdot \boldsymbol{T}(\phi, \boldsymbol{v}^0) - \boldsymbol{T}(\phi, \boldsymbol{u}) \cdot \partial_\varphi \boldsymbol{E}(\boldsymbol{v}^0) - \left\{ \boldsymbol{T}(\phi, \boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}^0) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big] \mathrm{d}x, \tag{4.10}$$

$$\langle \boldsymbol{g}^0_{\Omega\, b}, \boldsymbol{\varphi} \rangle_\Omega = \int_{\Omega(\phi)} \Big[\partial_\varphi (\boldsymbol{u} + \boldsymbol{v}^0) \cdot \boldsymbol{b} + \left\{ \boldsymbol{b} \cdot (\boldsymbol{u} + \boldsymbol{v}^0) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big] \mathrm{d}x. \tag{4.11}$$

If $\boldsymbol{p}(\phi)$ $\boldsymbol{u}_{\mathrm{D}}(\phi)$ and $\boldsymbol{C}(\phi)$ are fixed with material, we have

$$\left\langle \boldsymbol{g}^0_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} = \int_{\Gamma_p(\phi)} \Big[\partial_\varphi (\boldsymbol{u} + \boldsymbol{v}^0) \cdot \boldsymbol{p} + \boldsymbol{p} \cdot (\boldsymbol{u} + \boldsymbol{v}^0) \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot (\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu}) \right\} \Big] \mathrm{d}\gamma, \tag{4.12}$$

$$\langle \boldsymbol{g}^0_{\Gamma_{\mathrm{D}}}, \boldsymbol{\varphi} \rangle_{\Gamma_{\mathrm{D}}} = \int_{\Gamma_{\mathrm{D}}(\phi)} \Big[\partial_\varphi \boldsymbol{u} \cdot (\boldsymbol{T}(\phi, \boldsymbol{v}^0)\, \boldsymbol{\nu})$$

§4.2 Shape derivative of mean compliance (cnt.)

$$
\begin{aligned}
&+ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left\{ \boldsymbol{C}(\boldsymbol{\phi}) \, \partial_{\varphi} \boldsymbol{E}\left(\boldsymbol{v}^{0}\right) \boldsymbol{\nu} \right\} \\
&+ \left\{ \boldsymbol{C}(\boldsymbol{\phi}) \, \partial_{\varphi} \boldsymbol{E}(\boldsymbol{u}) \, \boldsymbol{\nu} \right\} \cdot \left(\boldsymbol{v}^{0} - \boldsymbol{u}_{\mathrm{D}}\right) + \left(\boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \, \boldsymbol{\nu}\right) \cdot \partial_{\varphi} \boldsymbol{v}^{0} \\
&+ \left\{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \left(\boldsymbol{T}\left(\boldsymbol{\phi}, \boldsymbol{v}^{0}\right) \boldsymbol{\nu}\right) + \left(\boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \, \boldsymbol{\nu}\right) \cdot \left(\boldsymbol{v}^{0} - \boldsymbol{u}_{\mathrm{D}}\right) \right\} \\
&\times \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu}\right) \right\} \Big] \, \mathrm{d}\gamma. \qquad (4.13)
\end{aligned}
$$

3. If $\boldsymbol{C}(\boldsymbol{\phi})$ and $\boldsymbol{b}(\boldsymbol{\phi})$ are varying with domain measure, considering that $\boldsymbol{u} = \boldsymbol{v}^{0}$ is fixed in space, we have

$$
\left\langle \boldsymbol{g}^{0}_{\Omega\, T}, \boldsymbol{\varphi} \right\rangle_{\Omega} = \int_{\Omega(\boldsymbol{\phi})} \left[-\partial_{\varphi} \boldsymbol{E}(\boldsymbol{u}) \cdot \boldsymbol{T}\left(\boldsymbol{\phi}, \boldsymbol{v}^{0}\right) - \boldsymbol{T}(\boldsymbol{\phi}, \boldsymbol{u}) \cdot \partial_{\varphi} \boldsymbol{E}\left(\boldsymbol{v}^{0}\right) \right] \mathrm{d}x, \qquad (4.14)
$$

$$
\left\langle \boldsymbol{g}^{0}_{\Omega\, b}, \boldsymbol{\varphi} \right\rangle_{\Omega} = \int_{\Omega(\boldsymbol{\phi})} \partial_{\varphi} \left(\boldsymbol{u} + \boldsymbol{v}^{0}\right) \cdot \boldsymbol{b} \, \mathrm{d}x. \qquad (4.15)
$$

§4.2 Shape derivative of mean compliance (cnt.)

If $\boldsymbol{p}(\phi)$ and $\boldsymbol{u}_{\mathrm{D}}(\phi)$ are varying with boundary measure, $\boldsymbol{C}(\phi)$ is varying with domain measure, we have

$$\left\langle \boldsymbol{g}^{0}_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} = \int_{\Gamma_p(\phi)} \partial_{\varphi}\left(\boldsymbol{u} + \boldsymbol{v}^{0}\right) \cdot \boldsymbol{p} \, \mathrm{d}\gamma, \tag{4.16}$$

$$\begin{aligned} \left\langle \boldsymbol{g}^{0}_{\Gamma_{\mathrm{D}}}, \boldsymbol{\varphi} \right\rangle_{\Gamma_{\mathrm{D}}} = \int_{\Gamma_{\mathrm{D}}(\phi)} \Big[& \partial_{\varphi}\boldsymbol{u} \cdot \left(\boldsymbol{T}\left(\phi, \boldsymbol{v}^{0}\right) \boldsymbol{\nu}\right) \\ & + \left(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}\right) \cdot \left\{\boldsymbol{C}(\phi)\, \partial_{\varphi}\boldsymbol{E}\left(\boldsymbol{v}^{0}\right) \boldsymbol{\nu}\right\} \\ & + \left\{\boldsymbol{C}(\phi)\, \partial_{\varphi}\boldsymbol{E}(\boldsymbol{u})\, \boldsymbol{\nu}\right\} \cdot \left(\boldsymbol{v}^{0} - \boldsymbol{u}_{\mathrm{D}}\right) + \left(\boldsymbol{T}(\phi, \boldsymbol{u})\, \boldsymbol{\nu}\right) \cdot \partial_{\varphi}\boldsymbol{v}^{0} \\ & - \left\{\boldsymbol{u}_{\mathrm{D}} \cdot \left(\boldsymbol{T}\left(\phi, \boldsymbol{v}^{0}\right) \boldsymbol{\nu}\right) + \left(\boldsymbol{T}(\phi, \boldsymbol{u})\, \boldsymbol{\nu}\right) \cdot \boldsymbol{u}_{\mathrm{D}}\right\} \\ & \times \left\{\boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu}\right)\right\} \Big] \mathrm{d}\gamma. \end{aligned} \tag{4.17}$$

§4.2 Shape derivative of mean compliance (cnt.)

4. If $\boldsymbol{p}(\phi) = p\boldsymbol{\nu}(\phi)$ is the static pressure, that is $p : D \to \mathbb{R}$ is fixed in space and $\boldsymbol{\nu}(\phi)$ is the normal of $\Omega(\phi)$, we have

$$\left\langle \boldsymbol{g}^0_{\Gamma_p}, \boldsymbol{\varphi} \right\rangle_{\Gamma_p} = \int_{\Gamma_p(\phi)} \Big[\partial_{\varphi} \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \cdot \boldsymbol{p} + \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \cdot \left\{ \left(\boldsymbol{\nabla} p \cdot \boldsymbol{\varphi} \right) \boldsymbol{\nu} + p \left(- \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \right) \boldsymbol{\nu} + \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu} \right) \boldsymbol{\nu} - \boldsymbol{\nabla} \boldsymbol{\nu}^{\mathrm{T}} \boldsymbol{\varphi} \right) \right\} + \left(\boldsymbol{u} + \boldsymbol{v}^0 \right) \cdot \left(p \boldsymbol{\nu} \right) \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{\nu} \right) \right\} \Big] \mathrm{d}\gamma. \tag{4.18}$$

§5 Numerical example

We used FreeFEM++ with trianguler finite elements of second order for linear elastic problem and first order for the H^1 gradient method.

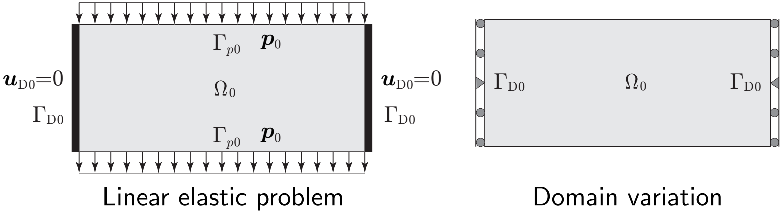


Fig. 15: Boundary conditions for shape optimization problem of plate both sides fixed

§5 Numerical example (cnt.)

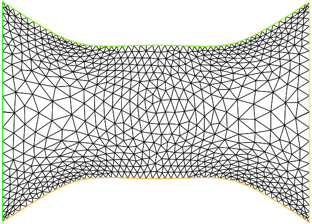


Fig. 16: Result of domain variation with $\boldsymbol{p}$ fixed in space

§5 Numerical example (cnt.)

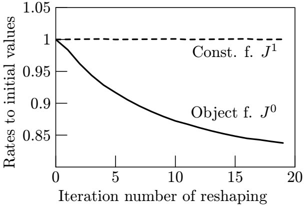


Fig. 17: Iteration history of cost functions with $\boldsymbol{p}$ fixed in space

§6 Conclusion

1. We defined variation rules with respect to domain variation of functions such as $\boldsymbol{b}(\phi)$, $\boldsymbol{p}(\phi)$, $\boldsymbol{u}_{\mathrm{D}}(\phi)$, $\boldsymbol{C}(\phi)$.
2. We showed furmulae required for computing the shape derivatives of functions applying the various variation rules.
   - Fixed in space
   - Fixed with material
   - Varying with domain measure
   - Varying with boundary measure
   - Static pressure
3. We demonstrated the results for shape derivatives of mean compliance in shape optimization problem of linear elastic body.

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