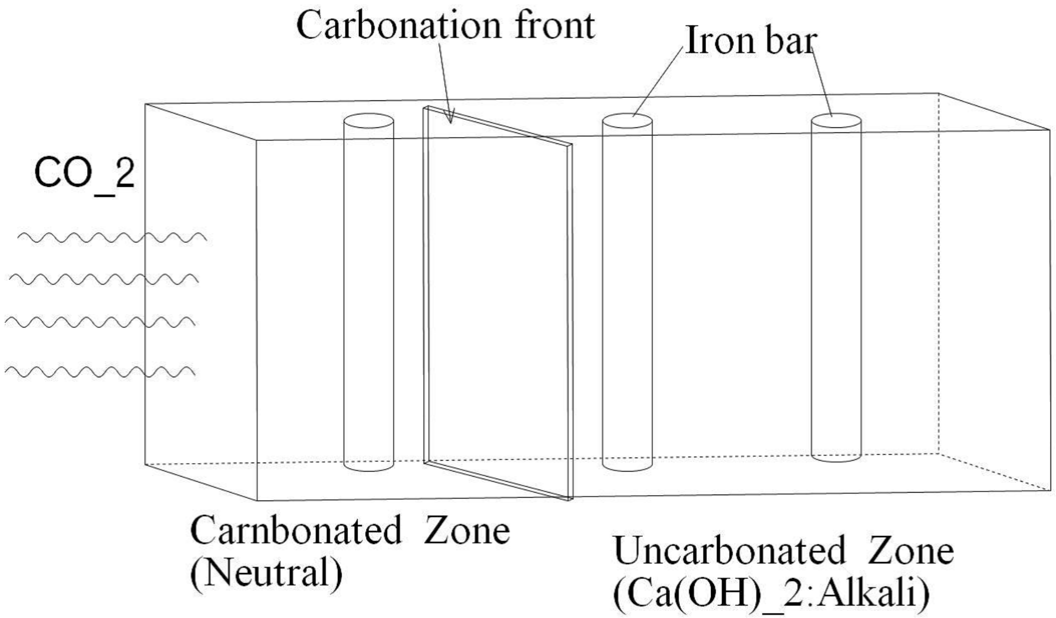
Recent results on mathematical models for concrete carbonation

AIKI, Toyohiko
Gifu University, Gifu, Japan
e-mail: aiki@gifu-u.ac.jp

a joint work with Dr. KUMAZAKI, Kota
Nagoya Institute of Technology, Nagoya, Japan

0. Introduction



Problems:

- Estimate the speed of carbonation

  Experimental law:

  $$\begin{pmatrix} \text{Position of} \\ \text{carbonation front} \end{pmatrix} \propto \sqrt{t}$$

  Aiki-Muntean (2011)

  ·1D model

  ·large time behavior of a free boundary

- Formulate 3D model with hysteresis (Aim of this talk)

Contents:

1. 3D model with hysteresis
2. Our results
3. Dual equation method

1. 3D model

Maekawa-Ishida-Kishi (2003) and Maekawa-Chaube-Kishi (1999)

(Moisture transport) and (Diffusion of CO_2)

1.1. Moisture transport

ρ_l: the density of water

ϕ: the porosity

S: the degree of saturation

h: relative humidity

C_l: positive constant

u: the concentration of CO_2 in water

w: the concentration of $Ca(OH)_2$ in water

$q_1 \geq 1, q_2 \geq 1$: constants

$\hat{K}$: the function of h

$\mathcal{H}$: the relative viscosity

$$\rho_l \frac{\partial}{\partial t}(\phi S) - \mathrm{div}\left[\left(\frac{\hat{K}(h)\phi^2}{\mathcal{H}} + C_l\phi(1-S)\right)\nabla h\right] = \phi S([u]^+)^{q_1}([w]^+)^{q_2}.$$

$\hat{K} : (0,1] \to \mathbf{R}$ is a continuous function such that $\displaystyle\lim_{h\downarrow 0}\frac{\hat{K}(h)}{h} = \infty$.

1.2 Hysteresis of saturation

Maekawa-Chaube-Kishi (1999, p.79)

$S = \mathcal{S}(h)$: a hysteresis operator with anticlockwise trend

1.3 Relative viscosity

Maekawa-Chaube-Kishi (1999, p.97)

$\eta = \mathcal{H}$ is a solution of ODE

$$\eta = a(1 + bH_d^2),$$

$$H_d'' + (\frac{1 + \eta''}{\eta})H_d = \frac{h}{\eta}.$$

1.4 Our model CP

$$\rho_l \frac{\partial}{\partial t}(\phi S) - \text{div}\left[\left(\frac{\widehat{K}(h)\phi^2}{\mathcal{H}} + C_l\phi(1-S)\right)\nabla h\right] = \phi S([u]^+)^{q_1}([w]^+)^{q_2}.$$

Here, we simplify the above problem

$$\phi = \text{ constant}, \quad S \propto h, \quad \mathcal{H} \propto h, \quad C_l = 0, \quad g(h) = \frac{\widehat{K}(h)}{Ch}\phi^2, \quad s = S.$$

We have:

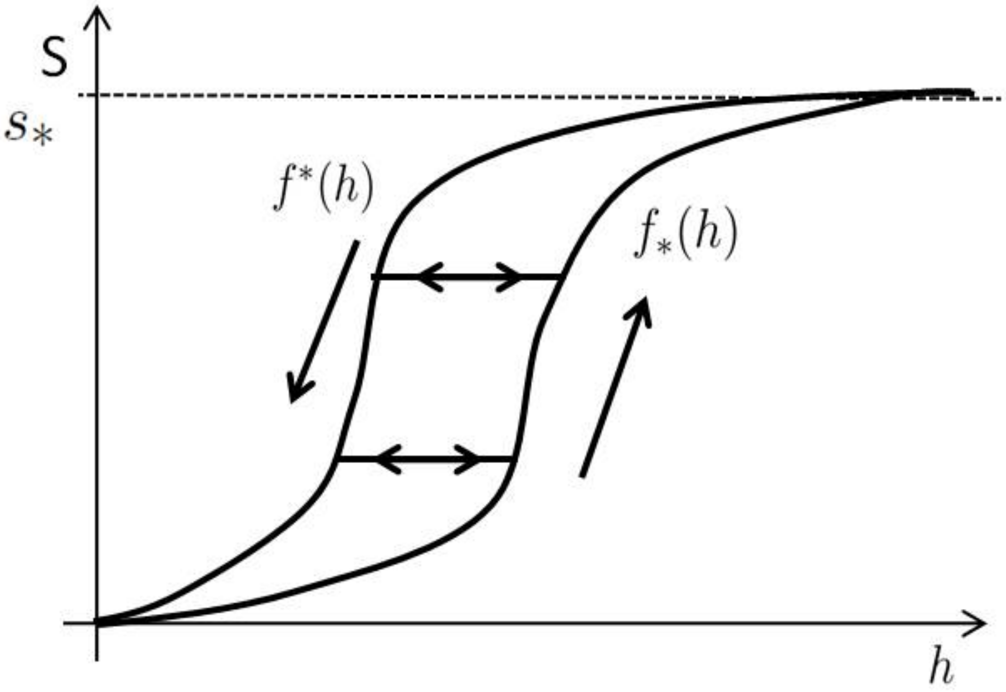
$$\rho h_t - \text{ div } (\nabla G(h)) = sf \quad \text{in } Q(T), \tag{1}$$

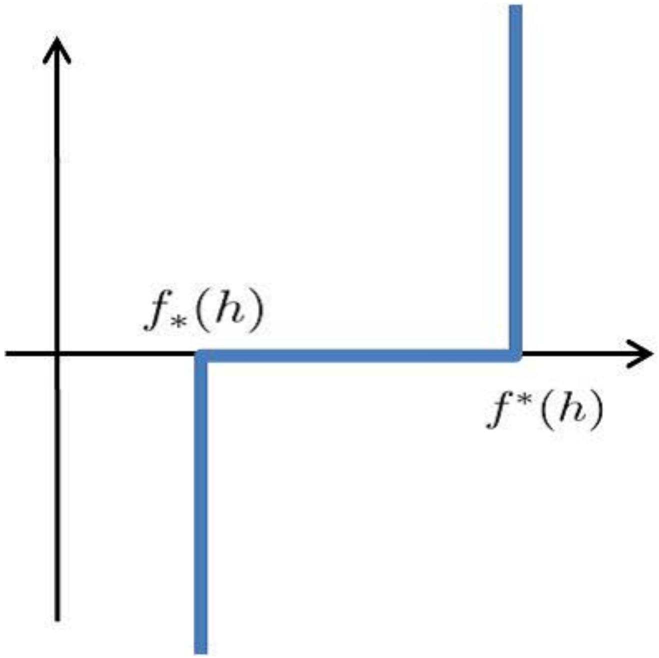
$$s_t - \nu \Delta s + \partial I(h; s) \ni 0 \quad \text{in } Q(T), \tag{2}$$

$$h = h_b, s = s_b \quad \text{on } S(T) := (0,T) \times \partial\Omega, \tag{3}$$

$$h(0) = h_0, s(0) = s_0 \quad \text{on } \Omega, \tag{4}$$

where $G(r) = \int_1^r g(s)ds$ for $r > 0$, $\nu > 0$, h_b and s_b are given functions on $S(T)$, h_0 and s_0 are initial functions, $Q(T) = (0,T) \times \Omega$, $\Omega \subset R^N$ is a bounded domain with the smooth boundary $\partial\Omega$ for $N \geq 1$.





$I(h;\cdot)$ is the indicator function of the interval $[f_*(h), f^*(h)]$, f_* and f^* are lower and upper curves indicated in the above figure, respectively.

$$\partial I(h;r) = \begin{cases} [0,\infty) & \text{for } r = f^*(h), \\ \{0\} & \text{for } f_*(h) < r < f^*(h), \\ (-\infty, 0] & \text{for } r = f_*(h). \end{cases}$$

$$s_t + \partial I(h;s) \ni 0.$$

For the compactness: Approximation: $s_t - \nu\Delta s + \partial I(h;s) \ni 0$.

1.5. Related results

1) Unsaturated flow through a porous medium Visintin (1993)

$$(p+S)_t - \nabla\cdot[\nabla p + k(S)] = f \text{ in } Q(T),$$
$$S_t + \partial I(p;S) \ni 0 \qquad \text{in } Q(T),$$

Kubo (2004): Global existence of a solution, no uniqueness

2) More realistic model

$$S_t - \nabla[\kappa(S)(\nabla p + g)] = 0, S = \mathcal{F}(p),$$

κ is a function, g is the gravity acceleration and $\mathcal{F}$ is a hysteresis operator.

Bagagiolo - Visintin (2000)
Existence of a weak solution, $\kappa \geq C > 0$, $\mathcal{F}$ is an approximation of the Preisach operator by using a dynamic relaxation term. No uniqueness.

Bagagiolo - Visintin (2004)
the existence of a weak solution with a generalized play operator $\mathcal{F}$. No uniqueness.

Kordulová (2011) : a Preisach operator $\mathcal{F}$ and a positive and monotonically increasing function $\kappa : R \to R$, and showed the existence of a solution. No uniqueness.

3) Colli-Kenmochi-Kubo (2001)

$$(cu + Lw)_t - \Delta u = f,$$
$$w_t - \kappa\Delta w + g(u,w) + \partial I(u;w) \ni 0.$$

If κ is sufficiently small, then a solution exists globally. The uniqueness holds in case $\kappa = 0$.

1.6. Difficulties

(Continuous property of the hysteresis operator)

Roughly speaking, if

$$s_{it} - \nu \Delta s_i + \partial I(h_i; s_i) \ni 0 \text{ in } Q(T), i = 1, 2,$$

then

$$|s_1 - s_2|_{L^\infty(Q(T))} \leq C |h_1 - h_2|_{L^\infty(Q(T))}.$$

(Nonlinearity)

Usually, we have: If

$$\rho h_{it} - \text{ div } (\nabla G(h_i)) = f_i \text{ in } Q(T),$$

then by using L^1-technique

$$|h_1 - h_2|_{L^\infty(0,T;L^1(\Omega))} \leq C |f_1 - f_2|_{L^1(Q(T))}.$$

In order to overcome these difficulties we apply the dual equation method:

$$|h_1 - h_2|_{L^\infty(Q(T))} \leq CT |f_1 - f_2|_{L^\infty(Q(T))}.$$

2. Our result

2.1. Assumptions

(A1) ρ is a positive constant.

(A2) $G : (0,\infty) \to R$ is continuous, $g(r) := G'(r)$ is continuous on $(0,\infty)$, $g \in C^2((0,\infty))$ and $g(r) \geq g_0$ for $r > 0$, where g_0 is a positive constant.

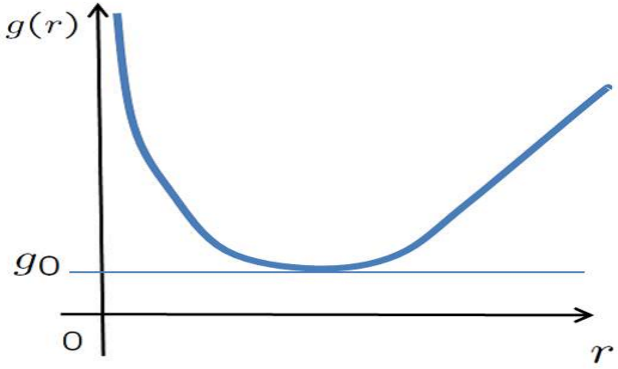
(A3) $f \in L^\infty(Q(T))$ with $f \geq 0$ a.e. on $Q(T)$.

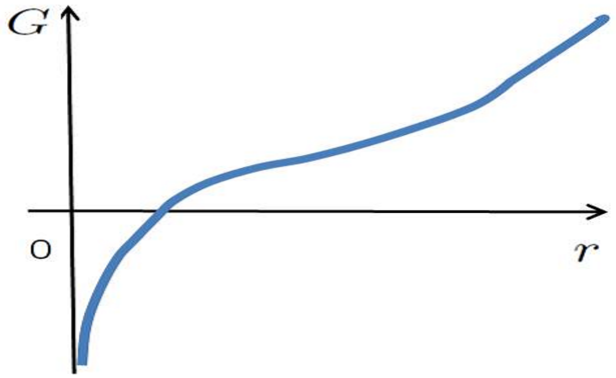
(A4) $f_*, f^* \in C^2(R) \cap W^{2,\infty}(R)$ with $0 \leq f_* \leq f^* \leq s_*$ on R, where s_* is a positive constant.

(A5) $u_b, w_b \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, $u_b \geq \kappa_0$ and $w_b \geq 0$ a.e. on $Q(T)$.

$u_0, w_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq \kappa_0$ and $w_0 \geq 0$ a.e. on Ω with $u_0 = u_b(0)$ and $w_0 = w_b(0)$ a.e. on $\partial\Omega$, where κ_0 is a positive constant.

Moreover, $f_*(u_0) \leq w_0 \leq f^*(u_0)$ a.e. on Ω and $f_*(u_b) \leq w_b \leq f^*(u_b)$ a.e. on $S(T)$.





$$\lim_{h \downarrow 0} \frac{\hat{K}(h)}{h} = \infty$$

$$g(h) = \frac{\hat{K}(h)}{Ch}\phi^2$$

2.2. Our Result

Let h and s be functions on $Q(T)$ for $0 < T < \infty$.
We call that the pair $\{h, s\}$ is a solution of CP on $[0, T]$,
if the conditions (S1) $\sim$ (S3) hold:
(S1) $h, s \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$
$h > 0$ a.e. on $Q(T)$.
(S2) (1), (3) and (4) hold.
(S3) There exists $\xi \in L^2(0, T; L^2(\Omega))$ such that
$\xi(t) \in \partial I(h(t); s(t))$ for a.e. $t \in [0, T]$, and

$$s_t - \nu \Delta s + \xi = 0 \quad \text{in } Q(T).$$

Theorem (Kumazaki-Aiki(2011)) If (A1) $\sim$ (A5) hold and $\nu \geq 0$, then CP has one and only one solution on $[0, T]$.

Existence: standard approximation method
Uniqueness: dual equation method Holmgren(1901)

3 Application of the dual equation method

f_i :given, $i = 1, 2$.

$$h_{it} - \Delta h_i = f_i \text{ in } Q(T).$$

Let $h = h_1 - h_2$ and $f = f_1 - f_2$.
Assume that η is a smooth function with $\eta(T) = 0$ and $\eta = 0$ on $(0, T) \times \partial\Omega$.
Then we have

$$-\int_{Q(T)} h(\eta_t + \Delta\eta)dxdt = \int_{Q(T)} f\eta dxdt.$$

Let ϕ be any smooth function.
Then there exists η such that

$$\begin{aligned} &\eta_t + \Delta\eta = \phi \text{ in } Q(T), \\ &\eta(T) = 0, \\ &\eta(t) = 0 \text{ a.e. on } \partial\Omega. \end{aligned}$$

Here, easily,

$$|\eta|_{L^1(Q(T))} \leq T|\phi|_{L^1(Q(T))}.$$

Hence,

$$\begin{aligned} & |\int_{Q(T)} \phi h dxdt| \\ = \ & |\int_{Q(T)} h(\eta_t + \Delta\eta)dxdt| \\ = \ & |\int_{Q(T)} f\eta dxdt| \\ \leq \ & |f|_{L^\infty(Q(T))}|\eta|_{L^1(Q(T))} \\ \leq \ & T|f|_{L^\infty(Q(T))}|\phi|_{L^1(Q(T))}. \end{aligned}$$

Therefore,

$$|h|_{L^\infty(Q(T))} \leq CT|f|_{L^\infty(Q(T))}.$$